

# New Linear Program Performance Bounds for Closed Queueing Networks<sup>1 2</sup>

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**Abstract.** We develop new linear program performance bounds for closed reentrant queueing networks based on an inequality relaxation of the average cost equation. The approach exploits the fact that the transition probabilities under certain policies of closed queueing networks are invariant within certain regions of the state space. This invariance suggests the use of a piecewise quadratic function as a surrogate for the differential cost function. The linear programming throughput bounds obtained are provably tighter than previously known bounds at the cost of increased computational complexity. Functional throughput bounds parameterized by the fixed customer population  $N$  are obtained, along with a bound

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on the limiting throughput as  $N \rightarrow +\infty$ . We show that one may obtain reduced complexity bounds while still retaining superiority.

**Key Words.** Performance Analysis, Queueing networks, Closed networks, Throughput.

## 1 Introduction

Linear programs (LPs) for bounding the throughput of a closed queueing network with a fixed customer population  $N$  were developed in [5, 2]. Also, in [5], bounds on the limiting throughput as the number of trapped customers increases to infinity were obtained. Linear programs to obtain a functional bound of the form  $\alpha^* N / (N + \nu)$  valid for all  $N$  were developed in [4]. Above  $\alpha^*$  is the maximum possible throughput sustainable for the network, and  $\nu$  is a bound on the asymptotic loss (defined in Section 2).

For open networks, [8] developed a procedure based on an inequality relaxation of the average cost equation. Here, we develop a procedure for LP performance bounds for closed queueing networks analogously to [8] but with some differences. The bounds exploit the key fact that the transition probabilities of a Markov chain modeling the closed network are invariant in certain regions of the state space. This invariance leads us to propose a piecewise quadratic function  $W(x)$  as a surrogate of the differential cost function in the invariant regions. The linear programs thereby obtained provide

a systematic method for determining the constant, linear, and quadratic coefficients of the surrogate  $W(x)$  in each region.

We also prove that the new upper and lower bounds on the throughput obtained by the approach are tighter than those of [5]. The ideas are also extended to obtain functional bounds of the form  $\alpha^*N/(N + \nu)$  by proposing a functional form for  $W(x)$ . It is seen that a bound on the limiting throughput may be extracted from the functional bounds. In all cases, the results are superior to those of [5] and [4].

The paper is organized as follows. The closed reentrant line queueing networks under consideration are described in Section 2, and the concepts of throughput, efficiency, and asymptotic loss are described. Section 3 recalls the average cost inequality which is the starting point for the development of the linear programming bounds, analogously to [8]. We develop the throughput bounds for fixed customer population  $N$  in Section 4. In Section 5, we propose a functional form for  $W(x)$  which leads to functional bounds on the throughput. Recognition of a certain structure in the functional bounds allows us to develop LPs to bound the limiting infinite population throughput. Procedures for reducing the complexity of the LPs while retaining some of the improvement in the bounds are considered in Section 6. We present concluding remarks in Section 7.

## 2 Description of Closed Reentrant Lines

A closed reentrant line is a queueing network consisting of  $S$  stations,  $\sigma_1, \dots, \sigma_S$ , at which customers receive service, and  $L$  buffers,  $b_1, \dots, b_L$ , at which customers await service, see Figure 1. Buffer  $b_i$  is served at station  $\sigma(i) \in \{\sigma_1, \dots, \sigma_L\}$ , and we write  $i \in \sigma$  to denote that  $\sigma(i) = \sigma$ . There is a fixed customer population of  $N$  trapped customers which circulate within the network along a closed deterministic route. After completing service from buffer  $b_i$  at station  $\sigma(i)$ , a customer moves next to buffer  $b_{i+1}$  at station  $\sigma(i+1)$  unless  $i = L$ , in which case the customer moves next to buffer  $b_1$  at station  $\sigma(1)$ . Equivalently, we interpret the buffer indices mod  $L$ , so that after service from  $b_i$ , a customer moves next to buffer  $b_{(i+1) \bmod L}$  at station  $\sigma((i+1) \bmod L)$ . The service time of a customer at buffer  $b_i$  is exponentially distributed with mean  $1/\mu_i > 0$ . We suppose that all service times are independent of each other. We call such a system a *closed reentrant line*, and note that the approach presented here may be extended to closed queueing networks with Bernoulli routing, as in [4].

Each station  $\sigma$  can serve only one customer at a time from those present in the buffers  $Buff(\sigma) = \{i : \sigma(i) = \sigma\}$  to which it caters. We also use the notation  $i \in \sigma$  to denote  $i \in Buff(\sigma)$ . A scheduling policy must be in place to direct the station in making its scheduling decisions. We assume that such a policy is both non-anticipative

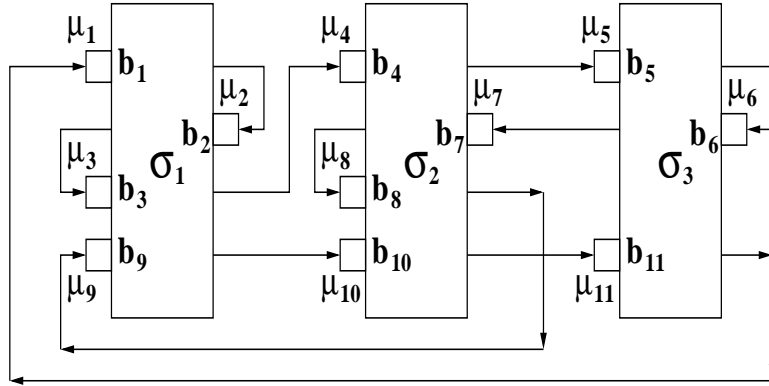


Figure 1: A closed reentrant line.

and non-idling, and we denote by  $\mathcal{U}$  the set of all such policies. By “non-idling” we mean that a station  $\sigma$  must provide service to some customer in a buffer in  $Buff(\sigma)$  if any customers are present. We also restrict attention to stationary policies, that is, policies dependent only upon the present locations of the  $N$  trapped customers.

We focus on buffer priority policies to demonstrate the ideas, and outline how to incorporate the class of all non-idling stationary policies. A buffer priority policy  $\theta$  is an element of  $\mathcal{U}$  which is specified by a permutation  $\theta = (\theta(1), \dots, \theta(L))$  of the buffer indices  $\{1, 2, \dots, L\}$ . Preemptive-resume priority is given to the non-empty buffer  $b_i$  at  $\sigma(i)$  for which  $\theta(i) < \theta(j)$  for all other non-empty buffers  $b_j \in Buff(\sigma(i))$ .

Now, consider a closed reentrant line operating under a scheduling policy  $u \in \mathcal{U}$  with a fixed customer population  $N$ . Let there be an initial condition describing the locations of the  $N$  trapped customers. Further, let  $D_L[0, T]$  denote the number of

departures from buffer  $b_L$  in the time interval  $[0, T]$ . We then call the random variable

$$\lambda_N^u := \liminf_{T \rightarrow +\infty} \frac{1}{T} D_L[0, T]$$

the *throughput* of the network.

Let  $\lambda^*$  be the *throughput capacity* of the line, that is,

$$\lambda^* = \text{Min}_\sigma \left\{ \frac{1}{\sum_{i \in \sigma} \frac{1}{\mu_i}} \right\}. \quad (1)$$

Clearly,  $\lambda_N^u \leq \lambda^*$  a.s. for all  $u \in \mathcal{U}$  and all customer populations  $N \geq 1$ . Any station achieving the “Min” in (1) is termed a *bottleneck* station. The line is called *balanced* if every station is a bottleneck.

For a buffer priority policy  $\theta$ , we say that the policy is *efficient* if

$$\lambda_\infty^\theta := \lim_{N \rightarrow +\infty} \lambda_N^\theta = \lambda^* \text{ a.s.}$$

(more precisely consider a sequence of scheduling policies, one for each  $N$ , with associated initial conditions). If the line is efficient for every policy  $u \in \mathcal{U}$ , i.e.,

$$\lambda_\infty^u := \liminf_{N \rightarrow +\infty} \lambda_N^u = \lambda^* \text{ a.s.}, \quad \forall u \in \mathcal{U},$$

then we say that the line is *guaranteed efficient*.

A finer measure of performance for a closed reentrant line under buffer priority policy  $\theta$  is the *asymptotic loss* defined as

$$\nu := \lim_{N \rightarrow +\infty} \frac{N(\lambda^* - \lambda_N^\theta)}{\alpha^*}$$

(more precisely one can consider the limsup denoted by  $\overline{\nu}$ , or the liminf denoted by  $\underline{\nu}$ ). Clearly, if the throughput is of the form  $\alpha^*N/(N + v)$ , then  $v$  is the asymptotic loss.

We suppose that all stochastic processes are right-continuous with left-limits. Let  $x_i(t)$  be the number of customers in buffer  $b_i$  at time  $t$ , and use  $x(t) = (x_1(t), \dots, x_L(t))^T$  to denote the location of the  $N$  customers at time  $t$ . We set  $w_i(t) = 1$  if a customer in buffer  $b_i$  is receiving service from station  $\sigma(i)$  at time  $t$  and 0 otherwise. Under a stationary policy  $u \in \mathcal{U}$ , or a buffer priority policy  $\theta$ ,  $\{X(t)\}$  is a finite-state time-homogeneous Markov chain, and we use  $w(x)$  to denote the scheduling decision in state  $x$ .

It is convenient to consider a sampled version of this process. We rescale time so that  $\sum_{i=1}^L \mu_i = 1$  and sample the system at all times  $\tau_n$  (with  $\tau_0 = 0$ ) at which either a real or a virtual service completion occurs. That is, we attach an exponential clock of rate  $\mu_i$  to each buffer and sample the system whenever a clock rings, resetting the clocks at such a time. Thus, if a customer in buffer  $b_i$  is receiving service when the  $i$ th clock rings, we consider that a real service has occurred and that a virtual service occurs otherwise which does not change the system state. Let  $x(n) := X(\tau_n)$  be the state of the process at the  $n$ th sampling time, and  $w(n) := w(\tau_n)$ . The sampled process  $\{x(n)\}$  is a finite-state discrete-time time-homogeneous Markov chain whose equilibrium distribution is the same as that of the original process. This sampling is

referred to as *uniformization* [6].

The sampled process  $\{x(n); n = 1, \dots\}$  has transition probabilities

$$P(x(n+1) = x(n) - e_i + e_{i+1} \mid x(n), w(n)) = \mu_i w_i(n),$$

where  $e_i$  denotes the vector of 0's with a 1 in the  $i$ th position,  $e$  the vector of all 1's, and  $e_{L+1} = e_1$ . For all other transitions, the probability is 0 for the reentrant line. Note that since the system state can remain unchanged after a virtual transition, the Markov chain is aperiodic. Also, under any buffer priority policy  $\theta$ , the chain has single closed communicating class, as the state  $x = (0, \dots, 0, N, 0, \dots, 0)^T$ , where the non-empty buffer is the lowest priority buffer at some station, is reachable from every state.

### 3 Average Cost Inequality

By relaxation of the average cost equation to inequality one obtains a bound on the performance. It is these inequalities that we will exploit to obtain LP performance bounds.

**Lemma 3.1. Average cost inequality performance bounds.** Consider a discrete-time discrete-state Markov chain with transition probability matrix  $P = [p_{x,y}]$ . Let  $c : \mathcal{S} \rightarrow \mathbb{R}$  be a cost function on the state space  $\mathcal{S}$  of the Markov chain and let  $W : \mathcal{S} \rightarrow \mathbb{R}$  be the surrogate of the differential cost function.



(i) Suppose there exists a  $J \in \mathbb{R}$  and a function  $W$ , bounded on  $\mathcal{S}$ , satisfying

$$J + W(x) \geq c(x) + \sum_{y \in \mathcal{S}} p_{x,y} W(y), \quad \forall x \in \mathcal{S}, \quad (2)$$

then

$$J \geq \limsup_{T \rightarrow \infty} \frac{1}{T} \sum_{n=0}^{T-1} E[c(x(n))]. \quad (3)$$

(ii) Suppose there exists a  $J \in \mathbb{R}$  and a function  $W$ , bounded on  $\mathcal{S}$ , satisfying

$$J + W(x) \leq c(x) + \sum_{y \in \mathcal{S}} p_{x,y} W(y), \quad \forall x \in \mathcal{S},$$

then

$$J \leq \liminf_{T \rightarrow \infty} \frac{1}{T} \sum_{n=0}^{T-1} E[c(x(n))]. \quad (4)$$

**Proof:**

(i) Since (2) holds for all  $x \in \mathcal{S}$ , in particular it holds for  $x = x(n)$ , so that

$$J + W(x(n)) \geq c(x(n)) + E[W(x(n+1)) \mid \mathcal{F}_n],$$

where  $\mathcal{F}_n$  is the  $\sigma$ -field generated by the past. Taking the unconditional expectation, we have

$$J + E[W(x(n))] \geq E[c(x(n))] + E[W(x(n+1))].$$

Summing from  $n = 0$  to  $n = T - 1$ , telescoping, and dividing by  $T$  we obtain

$$J + \frac{E[W(x(0))]}{T} \geq \frac{1}{T} \sum_{n=0}^{T-1} E[c(x(n))] + \frac{E[W(x(T))]}{T}.$$

Taking the limit as  $T \rightarrow \infty$  and applying the boundedness of  $W(x)$  over the finite state space, we obtain the result.

(ii) The lower bound is proved in the same manner. □

Thus, if we can find a  $J$  and a  $W$  for a specified  $c$ , we will have obtained an upper or lower bound on the long run average cost. The challenge in the following section will be to reduce these inequalities to a linear program for each fixed customer population level  $N$ .

## 4 Linear Program Throughput Bounds with Fixed Customer Population $N$

Consider first a closed reentrant line operating under a buffer priority policy  $\theta$ . Suppose there are  $N$  trapped customers circulating within the network, and let  $N \gg L$  (for simplicity, though it is not at all necessary). The state space is then

$$\mathcal{S} := \{x \in \mathbb{Z}_+^L : \sum_{i=1}^L x_i = N, x_i \geq 0\}.$$

Let  $\phi := \{\phi_1, \dots, \phi_L\}^T$ , where  $\phi_i = 0$  or  $1$ . Each such  $\phi$  labels a region of the state space

$$\mathcal{X}^\phi := \{x \in \mathcal{S} : x_i = 0 \text{ if } \phi_i = 0, x_i \geq 1 \text{ if } \phi_i = 1\},$$

in which the transition probabilities are identical for all  $x \in \mathcal{X}^\phi$ . Note that there are  $2^L - 1$  possible  $\phi$ 's, as  $\phi = (0, \dots, 0)^T$  is not possible. Clearly, the scheduling decision  $w(x)$  is constant over all  $x \in \mathcal{X}^\phi$  under a buffer priority policy  $\theta$ . For each such  $x$ , let  $\phi(x)$  denote the label  $\phi$  for which  $x \in \mathcal{X}^\phi$ .

On each region  $\mathcal{X}^\phi$ , we propose a separate quadratic form  $W^\phi$ ,

$$W^\phi(x) := c^\phi + x^T e d^\phi + x^T e e^T x f^\phi + r^{\phi^T} x + x^T e p^{\phi^T} x + \frac{1}{2} x^T Q^\phi x,$$

and consider the composite

$$W(x) := W^{\phi(x)}, \forall x \in \mathcal{X}^\phi.$$

This composite function will serve as the surrogate for the differential cost function. Above  $x^T e = N$ , by our assumption on the total customer population. With this piecewise quadratic function, we consider how to verify the inequalities of (2) for the cost function

$$c(x) = \frac{1}{L} \sum_{\{i:w_i(x)=1\}} \mu_i,$$

so that  $J$  will be an upper bound on the throughput.

We introduce a further partition of  $\mathcal{S}$ . Let  $\psi := \{\psi_1, \dots, \psi_L\}^T$  be a vector with  $\psi_i \in \{0, 1, 2\}$ . Each  $\psi$  labels a region of the state space  $\mathcal{S}$ ,

$$\mathcal{Y}^\psi := \{x \in \mathcal{S} : x_i = 0 \text{ if } \psi_i = 0, x_i = 1 \text{ if } \psi_i = 1, x_i \geq 2 \text{ if } \psi_i = 2\}.$$

It is easy to see that  $\mathcal{Y}^\psi$  is a subset of  $\mathcal{X}^{\phi(\psi)}$ . Notice also that  $\phi(x)$  does not vary with  $x$  unless the set of non-zero components is changed. Thus, if  $x_i = 1$  or  $x_{i+1} = 0$ , then  $\phi(x - e_i + e_{i+1})$  is different from  $\phi(x)$ . Otherwise,  $\phi(x - e_i + e_{i+1}) = \phi(x)$ .

Throughout, we use the notation  $I(\phi)$  to denote the set of buffers receiving service when  $x \in \mathcal{X}^\phi$ , i.e.,  $I(\phi) = \{i : w_i(\phi) = 1\}$ . Also, recall that  $\phi(\psi)$  is the label  $\phi$  for which  $\psi \in \mathcal{X}^\phi$ .

Now consider  $x \in \mathcal{Y}^\psi$ . The inequalities of (2) that we seek to satisfy are

$$\begin{aligned}
J \geq \sum_{i \in I(\phi(x))} \mu_i \left[ \frac{1}{L} \right. &+ c^{\phi(x-e_i+e_{i+1})} + Nd^{\phi(x-e_i+e_{i+1})} + N^2 f^{\phi(x-e_i+e_{i+1})} \\
&+ r^{\phi(x-e_i+e_{i+1})T} (x - e_i + e_{i+1}) + Np^{\phi(x-e_i+e_{i+1})T} (x - e_i + e_{i+1}) \\
&+ \frac{1}{2}(x - e_i + e_{i+1})^T Q^{\phi(x-e_i+e_{i+1})} (x - e_i + e_{i+1}) \\
&- c^{\phi(x)} - Nd^{\phi(x)} - N^2 f^{\phi(x)} \\
&\left. - r^{\phi(x)T} x - Np^{\phi(x)T} x - \frac{1}{2}x^T Q^{\phi(x)} x \right], \forall x \in \mathcal{Y}^\psi,
\end{aligned}$$

for all  $\psi$ . Since for all  $x \in \mathcal{Y}^\psi$  the set of non-zero components of the state vector is the same, the labels  $\phi$  in the above are dependent only upon  $\psi$  and not  $x$ . Thus, the inequalities become

$$\begin{aligned}
J \geq \sum_{i \in I(\phi(\psi))} \mu_i \left[ \frac{1}{L} \right. &+ c^{\phi(\psi-e_i+e_{i+1})} + Nd^{\phi(\psi-e_i+e_{i+1})} + N^2 f^{\phi(\psi-e_i+e_{i+1})} \\
&+ r^{\phi(\psi-e_i+e_{i+1})T} (x - e_i + e_{i+1}) + Np^{\phi(\psi-e_i+e_{i+1})T} (x - e_i + e_{i+1}) \\
&+ \frac{1}{2}(x - e_i + e_{i+1})^T Q^{\phi(\psi-e_i+e_{i+1})} (x - e_i + e_{i+1})
\end{aligned}$$

$$\begin{aligned}
& - c^{\phi(\psi)} - Nd^{\phi(\psi)} - N^2 f^{\phi(\psi)} \\
& - \left[ r^{\phi(\psi)T} x - Np^{\phi(\psi)T} x - \frac{1}{2}x^T Q^{\phi(\psi)} x \right], \forall x \in \mathcal{Y}^\psi,
\end{aligned}$$

for all  $\psi$ .

For each  $x \in \mathcal{Y}^\psi$ , we can write  $x = z + \psi$ , where  $z$  is a vector element of the set

$$\mathcal{S}_{N-|\psi|}^\psi := \{z \in \mathbb{Z}_+^L : z_i = 0 \text{ if } \psi = 0 \text{ or } 1, \sum_{\{i:\psi_i=2\}} z_i = N - |\psi|\}.$$

Above  $|\psi|$  is the sum of the components of  $\psi$ , i.e., the  $l_1$ -norm. We can rewrite the inequalities as (using  $N = x^T e = e^T x = (z + \psi)^T e = e^T (z + \psi)$ )

$$\begin{aligned}
J \geq \sum_{i \in I(\phi(\psi))} \mu_i \left[ \frac{1}{L} \right. & + c^{\phi(\psi - e_i + e_{i+1})} + e^T (z + \psi) d^{\phi(\psi - e_i + e_{i+1})} \\
& + f^{\phi(\psi - e_i + e_{i+1})} (z + \psi)^T e e^T (z + \psi) \\
& + r^{\phi(\psi - e_i + e_{i+1})T} (z + \psi - e_i + e_{i+1}) \\
& + (z + \psi)^T e p^{\phi(\psi - e_i + e_{i+1})T} (z + \psi - e_i + e_{i+1}) \\
& + \frac{1}{2} (z + \psi - e_i + e_{i+1})^T Q^{\phi(\psi - e_i + e_{i+1})} (z + \psi - e_i + e_{i+1}) \\
& - c^{\phi(\psi)} - e^T (z + \psi) d^{\phi(\psi)} - f^{\phi(\psi)} (z + \psi)^T e e^T (z + \psi) \\
& - r^{\phi(\psi)T} (z + \psi) - e^T (z + \psi) p^{\phi(\psi)T} (z + \psi) \\
& \left. - \frac{1}{2} (z + \psi)^T Q^{\phi(\psi)} (z + \psi) \right], \forall z \in \mathcal{S}_{N-|\psi|}^\psi,
\end{aligned}$$

for all  $\psi$ .

These inequalities contain only constant, linear, or quadratic terms in  $z$ . Collecting

such terms, we may write the above as

$$J \geq k^\psi + s^{\psi^T} z + \frac{1}{2} z^T M^\psi z, \forall z \in \mathcal{S}_{N-|\psi|}^\psi, \forall \psi,$$

for appropriate  $k^\psi, s^\psi$ , and symmetric  $M^\psi$  (provided in Appendix A.1) which are linear in the variables  $c^\phi, d^\phi, f^\phi, r^\phi, p^\phi$ , and  $Q^\phi$ . To reduce these quadratic inequalities to linear constraints, we impose the further requirements

$$0 \geq M^\psi \text{ (componentwise)}, \forall \psi, \text{ and} \tag{5}$$

$$J \geq k^\psi + s_i^\psi (N - |\psi|), \forall \psi, \forall i \text{ with } \psi_i = 2. \tag{6}$$

These further requirements are not necessary, though general nonlinear constraints are unwieldy for fixed  $N$ . An alternate approach to (5) would be to require

$$M^\psi \text{ conegative, for all } \psi,$$

where a matrix  $M$  is termed *conegative* if  $x^T M x \leq 0$  for all  $x \in \mathbb{Z}_+^L$  (testing for copositivity or conegativity is NP-Complete, see [3, 1]). Thus, we have arrived at the following linear program to bound the throughput of a closed reentrant line.

**Theorem 4.1. Linear program performance bounds for fixed  $N$ .** Consider a closed reentrant line operating under a buffer priority policy  $\theta$ .

(i) Let  $\{J, c^\phi, d^\phi, f^\phi, r^\phi, p^\phi$ , and symmetric  $Q^\phi\}$  be the decision variables in the linear program

$$\text{Min } J$$

subject to (5,6). The value of the linear program is an upper bound on the throughput  $\lambda_N^\theta$  as in (3).

(ii) Let  $\{J, c^\phi, d^\phi, f^\phi, r^\phi, p^\phi, \text{ and symmetric } Q^\phi\}$  be the decision variables in the linear program

$$\text{Max } J$$

subject to

$$\begin{aligned} 0 &\leq M^\psi \text{ (componentwise), } \forall \psi, \text{ and} \\ J &\leq k^\psi + s_j^\psi(N - |\psi|), \forall \psi, \forall j \text{ with } \psi_j = 2. \end{aligned}$$

The value is a lower bound as in (4).

(iii) The performance bounds of (i) and (ii) are at least as tight as those of [5].

**Proof:** (i) The application of Lemma 3.1 suffices, as  $W(x)$  is finite for all  $x \in \mathcal{S}$  (the state space itself is finite).

(ii) The same argument as in (i) applies.

(iii) The proof is given in Appendix A.2. □

**Example 4.2. Throughput bounds for fixed  $N$ .** Consider the closed reentrant line of Figure 2, with  $\mu_1 = 100/3, \mu_2 = 100/6, \mu_3 = 100/3, \mu_4 = 100/6$ . The buffer

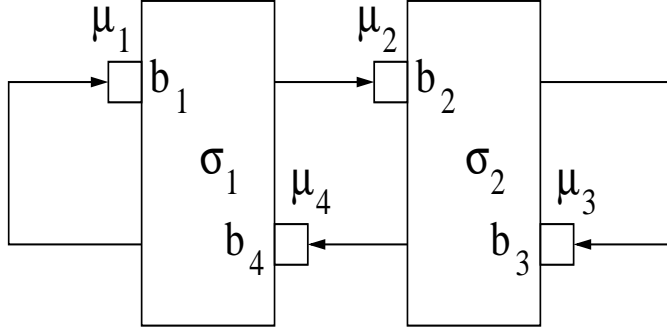


Figure 2: System of Example 4.2

priority policy in effect is  $\theta = \{4, 2, 3, 1\}$ , which gives priority to  $b_4$  at station  $\sigma_1$  and priority to  $b_2$  at station  $\sigma_2$ . In fact, this buffer priority policy induces a phenomenon known as a virtual station [7]. In essence, buffers  $\{b_2, b_4\}$  behave as another station (though they reside at separate stations) inducing an additional throughput constraint along with those induced by the actual stations themselves. This behavior arises from the fact that  $x_2 x_4 = 0$  in equilibrium, so that we may exclude from consideration any  $\psi$  for which  $\psi_2 \psi_4 \neq 0$  (such  $\mathcal{Y}^\psi$  are transient). Similarly, one may incorporate this behavior into the LPs of [5], as in [9]. The bounds of Figure 3 were obtained by solving a separate LP for each value of  $N$  considered. The bounds obtained via the approach developed here restricted attention to non-transient  $\psi$ , and the bounds of previous approaches incorporated the constraints induced by the virtual station. The improvement is evident.  $\square$

For the class of all non-idling policies, we may develop a similar procedure as for open networks [8]. One merely accounts for every feasible transition in each region  $\mathcal{X}^\phi$ ,



Throughput Bounds for Network of Fig. 2	Trapped Customer Population = 10	Trapped Customer Population = 20	Trapped Customer Population = 100
Upper Throughput Bound of [4]	8.333	8.333	8.333
New Upper Throughput Bound	7.936	8.130	8.292
New Lower Throughput Bound	7.476	7.584	7.671
Lower Throughput Bound of [4]	6.536	6.601	6.653

Figure 3: Throughput bounds for the system of Example 4.2.

proposing a piecewise quadratic function  $W(x) := W^\phi(x), \forall x \in \mathcal{X}^\phi$  as the surrogate of the differential cost function. Thus, we merely obtain more constraints.

To incorporate virtual multiserver stations [7], as in Example 4.2, one eliminates transient facets of the state space from consideration [8]. Similarly, if one can identify other regions of the state space  $\mathcal{S}$  which are transient, they too should be eliminated from consideration (e.g., successive reentrancy [8]). In either case, the elimination of

states from  $\mathcal{S}$  results in improved linear program bounds.

## 5 Functional Bounds and the Limiting Infinite Population LPs

Rather than solving a separate LP for each fixed customer population  $N$ , it is desirable to obtain a bound parameterized by  $N$ , that is, a functional bound. This has been done in [4], and we will develop functional bounds via the average cost inequality which are provably tighter than those obtained in [4]. The idea is to propose a general functional form for the surrogate of the differential cost function.

Let

$$\begin{aligned} J(N) &= \frac{\alpha'N}{N + \nu}, \text{ and} \\ W^\phi(x, N) &= \frac{W^\phi(x)}{N + \nu}, \end{aligned}$$

where  $W^\phi(x)$  is as in Section 4, and consider the composite

$$W(x, N) = W^\phi(x, N), \forall x \in \mathcal{X}^\phi.$$

For a buffer priority policy  $\theta$ , the inequalities of (2) take the form

$$\begin{aligned} \frac{\alpha'N}{N + \nu} \geq & \sum_{i \in I(\phi(\psi))} \mu_i \left[ \frac{1}{L} + \frac{c^{\phi(\psi - e_i + e_{i+1})} + d^{\phi(\psi - e_i + e_{i+1})} e^T x}{N + \nu} \right. \\ & + \frac{x^T e e^T x f^{\phi(\psi - e_i + e_{i+1})}}{N + \nu} \\ & \left. + \frac{r^{\phi(\psi - e_i + e_{i+1})} (x - e_i + e_{i+1})}{N + \nu} \right] \end{aligned}$$

$$\begin{aligned}
& + \frac{x^T e p^{\phi(\psi - e_i + e_{i+1})^T} (x - e_i + e_{i+1})}{N + \nu} \\
& + \frac{1}{2} \frac{(x - e_i + e_{i+1})^T Q^{\phi(\psi - e_i + e_{i+1})} (x - e_i + e_{i+1})}{N + \nu} \\
& - \left( \frac{c^{\phi(\psi)} + d^{\phi(\psi)} e^T x + x^T e e^T x f^{\phi(\psi)}}{N + \nu} \right) \\
& - \left( \frac{r^{\phi(\psi)^T} x + x^T e p^{\phi(\psi)^T} x}{N + \nu} \right) \\
& - \left. \frac{1}{2} \frac{x^T Q^{\phi(\psi)} x}{N + \nu} \right], \forall x \in \mathcal{Y}^\psi, \forall \psi.
\end{aligned}$$

If  $N + \nu > 0$ , we can multiply through by  $N + \nu$  without disturbing the inequality.

Further, substituting  $x = z + \psi$  for  $x \in \mathcal{Y}^\psi$  we equivalently seek to satisfy

$$\begin{aligned}
\alpha' N \geq \sum_{i \in I(\phi(\psi))} \mu_i \left[ \frac{\nu}{L} \right. & + \frac{(z + \psi)^T e}{L} + c^{\phi(\psi - e_i + e_{i+1})} + e^T (z + \psi) d^{\phi(\psi - e_i + e_{i+1})} \\
& + f^{\phi(\psi - e_i + e_{i+1})} (z + \psi)^T e e^T (z + \psi) \\
& + r^{\phi(\psi - e_i + e_{i+1})^T} (z + \psi - e_i + e_{i+1}) \\
& + (z + \psi)^T e p^{\phi(\psi - e_i + e_{i+1})^T} (z + \psi - e_i + e_{i+1}) \\
& + \frac{1}{2} (z + \psi - e_i + e_{i+1})^T Q^{\phi(\psi - e_i + e_{i+1})} (z + \psi - e_i + e_{i+1}) \\
& - c^{\phi(\psi)} - e^T (z + \psi) d^{\phi(\psi)} - f^{\phi(\psi)} (z + \psi)^T e e^T (z + \psi) \\
& - r^{\phi(\psi)^T} (z + \psi) - e^T (z + \psi) p^{\phi(\psi)^T} (z + \psi) \\
& \left. - \frac{1}{2} (z + \psi)^T Q^{\phi(\psi)} (z + \psi) \right],
\end{aligned}$$

for all  $z \in \mathcal{S}_{N-|\psi|}^\psi$  and for all  $\psi$ . These inequalities consist of constant, linear, and

quadratic terms in  $z$ . Grouping these terms we may rewrite the above inequalities as

$$\alpha' N \geq \sum_{i \in I(\phi(\psi))} \mu_i \left( \frac{\nu}{L} \right) + l^\psi + t^{\psi^T} z + \frac{1}{2} z^T R^\psi z, \forall z \in \mathcal{S}_{N-|\psi|}^\psi, \forall \psi,$$

for appropriate  $l^\psi, t^\psi$ , and symmetric  $R^\psi$  (provided in Appendix A.3) which are linear in  $c^\phi, d^\phi, f^\phi, r^\phi, p^\phi$ , and  $Q^\phi$ .

We seek to enforce these inequalities for all  $N$ , thereby ensuring that the functional form proposed in fact holds for each  $N$  such that  $N + \nu > 0$ . In order to obtain constraints arising from these inequalities we may allow nonlinear constraints or require linear ones. The following Lemma is proved in [8] and shows how one may obtain nonlinear constraints which are necessary and sufficient for nonnegativity of a quadratic form on an orthant.

**Lemma 5.1. Nonnegativity of quadratic forms in the positive orthant.** A quadratic  $c + p^T x + \frac{1}{2} x^T Q x$  is nonnegative over all  $x \in \mathcal{R}_+^L$  if and only if

$$c \geq 0$$

$Q$  is copositive, and

$$x^T (cQ - pp^T / 2) x \geq 0, \forall x \in \mathcal{R}_+^L \text{ with } p^T x < 0.$$

Thus, one may formulate a nonlinear program to obtain feasible values for  $c^\phi, d^\phi, f^\phi, r^\phi, p^\phi$ , and  $Q^\phi$  so that the inequalities arising from (2) are satisfied. As we are primarily interested in linear programming bounds for which a minimum value is

more readily obtained, we proceed with the goal of developing LP bounds as follows.

For each  $N$ , we require

$$0 \geq R^\psi \text{ (componentwise), } \forall \psi, \text{ and}$$

$$\alpha' N \geq \sum_{i \in I(\phi(\psi))} \mu_i \left( \frac{\nu}{L} \right) + l^\psi + t_j^\psi (N - |\psi|), \forall j \text{ with } \psi_j, \forall \psi.$$

Since we seek a functional bound viable for all  $N > -\nu$  (recall that we are in fact only interested in  $N > 0$ ), we may require

$$0 \geq R^\psi \text{ (componentwise), } \forall \psi, \tag{7}$$

$$\alpha' \geq t_j^\psi, \forall j \text{ with } \psi_j = 2, \forall \psi, \text{ and} \tag{8}$$

$$0 \geq l^\psi - (\psi^T e) t_j^\psi + \sum_{i \in I(\phi(\psi))} \mu_i \left( \frac{\nu}{L} \right), \forall j \text{ with } \psi_j = 2, \forall \psi. \tag{9}$$

Prior to obtaining the functional bounds, we develop the limiting infinite population LP bound. Observe that  $\nu$  occurs only in the constraints of (9) for all  $\psi$ . Thus, if  $\alpha', c^\phi, d^\phi, f^\phi, r^\phi, p^\phi$ , and symmetric  $Q^\phi$  are feasible for (7,8) for all  $\psi$ , then a finite  $\nu$  may be chosen to satisfy (9) for all  $\psi$ . Thus, for any  $N > -\nu$ , the feasibility of (7,8,9) implies that  $\alpha' N / (N + \nu)$  is in fact an upper bound on the throughput  $\lambda_N^\theta$  for all such  $N$ . Hence,  $\alpha'$  is an upper bound on the limiting throughput as  $N \rightarrow \infty$ , and we have the following theorem via an application of Lemma 3.1. The lower bound is developed similarly.

**Theorem 5.2. The limiting infinite population linear program.** Consider a closed reentrant line operating under a buffer priority policy  $\theta$ .

(i) Let  $\{\alpha, c^\phi, d^\phi, f^\phi, r^\phi, p^\phi$ , and symmetric  $Q^\phi\}$  be the decision variables in the linear

program  $\overline{T}_\infty^\theta$

Min  $\alpha$

subject to (7,8) for all  $\psi$ . The value of the linear program  $V\overline{T}_\infty^\theta$  is an upper bound on the limiting throughput  $\lambda_\infty^\theta$  as in (3).

(ii) Let  $\{\alpha, c^\phi, d^\phi, f^\phi, r^\phi, p^\phi$ , and symmetric  $Q^\phi\}$  be the decision variables in the linear

program  $\underline{T}_\infty^\theta$

Max  $\alpha$

subject to

$$0 \leq R^\psi \text{ (componentwise), } \forall \psi, \text{ and}$$

$$\alpha' \leq t_j^\psi, \forall j \text{ with } \psi_j = 2, \forall \psi.$$

The value of the linear program  $V\underline{T}_\infty^\theta$  is a lower bound on the limiting throughput  $\lambda_\infty^\theta$  as in (4).

(iii) The limiting throughput bounds  $V\underline{T}_\infty^\theta$  and  $V\overline{T}_\infty^\theta$  are at least as tight as those of [4].

The proofs of (i) and (ii) are apparent from the development preceding the theorem. The proof of (iii) is similar to that of Theorem 4.1 (iii) which is provided in Appendix A.2.

Now, following the above arguments, one may obtain functional bounds for the throughput  $\lambda_N^\theta$ . Utilizing the value of the LPs of Theorem 5.2 as the asymptotes for the functional bounds, we have the following theorem.

**Theorem 5.3. The functional bound linear programs.** Consider a closed reentrant line operating under a buffer priority policy  $\theta$ .

(i) Let  $\overline{FB}^\theta$  denote the linear program

$$\text{Max } \nu$$

subject to (7,9) and

$$V\overline{T}_\infty^\theta \geq t_j^\psi, \forall j \text{ with } \psi_j = 2, \forall \psi.$$

Let the value of this linear program be  $V\overline{FB}^\theta$ . For all  $N + V\overline{FB}^\theta > 0$ ,

$$\lambda_N^\theta \leq \frac{N V\overline{T}_\infty^\theta}{N + V\overline{FB}^\theta}.$$

(ii) Let  $\underline{FB}^\theta$  denote the linear program

$$\text{Min } \nu$$

subject to

$$\begin{aligned}
0 &\leq R^\psi, \forall \psi, \\
V\underline{T}_\infty^\theta &\leq t_j^\psi, \forall j \text{ with } \psi_j = 2, \forall \psi, \text{ and} \\
0 &\leq l^\psi - (\psi^T e)t_j^\psi + \sum_{i \in I(\phi(\psi))} \mu_i \left( \frac{\nu}{L} \right), \forall j \text{ with } \psi_j = 2, \forall \psi.
\end{aligned}$$

Let the value of this linear program be  $V\underline{FB}^\theta$ . For all  $N + V\underline{FB}^\theta > 0$ ,

$$\frac{N V\underline{T}_\infty^\theta}{N + V\underline{FB}^\theta} \leq \lambda_N^\theta.$$

- (iii) Let  $\underline{\alpha}$  and  $\overline{\alpha}$  be the optimal solutions to the limiting infinite population LPs of [4] for lower and upper throughput bounds, respectively. With  $\underline{\alpha}$  and  $\overline{\alpha}$  replacing the values  $V\underline{T}_\infty^\theta$  and  $V\overline{T}_\infty^\theta$  respectively in the above LPs, the bounds on the asymptotic loss  $\underline{FB}^\theta$  and  $\overline{FB}^\theta$  provide functional throughput bounds at least as tight as those of [4].

The proofs of Theorem 5.3 (i) and (ii) follow by application of Lemma 3.1. The proof of Theorem 5.3 (iii) is provided in Appendix A.4.

**Example 5.4. Limiting infinite population throughput and functional bounds.** Consider the network of Figure 4. The service rates are as in Example 4.2, that is  $\mu_1 = 100/3, \mu_2 = 100/6, \mu_3 = 100/3, \mu_4 = 100/6$ . We also consider the same buffer priority policy  $\theta = \{4, 2, 3, 1\}$ , so that the buffers  $b_2$  and  $b_4$  again



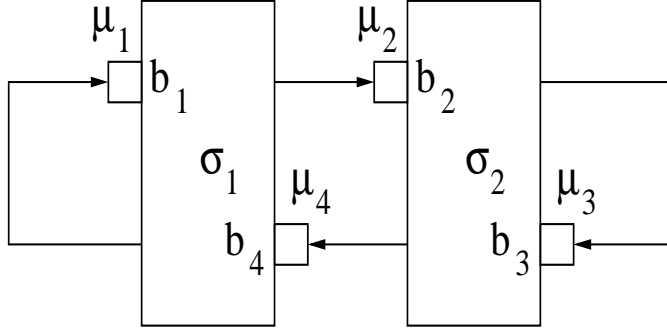


Figure 4: System of Example 5.4.

form a virtual station. Application of the results of this section, eliminating transient facets from consideration, yields functional throughput bounds. Similarly, as in [9], one may incorporate the existence of the virtual station into the functional bound LPs of [4]. The bounds thereby obtained are provided in Figure 5 and Figure 6. The dashed lines are the bounds of [4] accounting for the virtual station as in [9]. The solid lines are those of this paper. Again, the new bounds are tighter.  $\square$

As for the fixed customer population bounds, the approach is readily extended to the class of all non-idling policies by considering all feasible transitions in a given facet. Similarly, one may eliminate transient facets to improve the bounds.

## 6 Controlled Complexity Bounds

The number of regions which must be considered to obtain bounds via the methods presented in Sections 4 and 5 is on the order of  $3^L$ . Here we consider how one may reduce the complexity of the resulting LPs. First, we propose a single quadratic

Functional Throughput Bound	Analytic Form
Upper Bound of [4]	$\frac{100}{12}$
New Upper Bound	$\frac{100 N}{12 (N + 46/221)}$
New Lower Bound	$\frac{100 N}{13 (N + 1)}$
Lower Bound of [4]	$\frac{100 N}{15 (N + 1/5)}$

Figure 5: Analytic form of the functional throughput bounds for Example 5.4.

function  $W(x)$  for all  $x \in \mathcal{S}$  and develop the LPs by consideration of the regions  $\mathcal{X}^\phi$ . As a second alternative, we allow multiple pieces for  $W(x)$  on certain regions  $\mathcal{X}^\phi$  and consider  $\mathcal{Y}^\psi$  as necessary and  $\mathcal{X}^\phi$  otherwise. Third, we allow very simple composite quadratic functions and collapse the inequalities by extending the size of the simplex under consideration.

## 6.1 A single quadratic

It is not difficult to see, following the development of Section 4, that the use of a single quadratic function  $W(x) := c + Nd + N^2 f + r^T x + Np^T x + (1/2)x^T Qx$  for all  $x \in \mathcal{S}$  allows one to partition the state space into the regions  $\mathcal{X}^\phi$ . The further

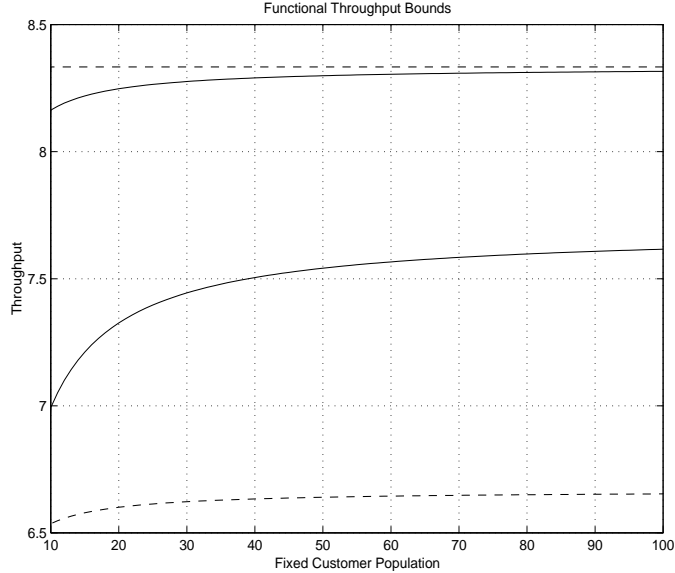


Figure 6: Functional throughput bounds for the network of Example 5.4.

partition  $\mathcal{Y}^\psi$  is extraneous. Analogously to Section 4, for each  $x \in \mathcal{X}^\phi$ , we can write  $x = z + \phi$ , where  $z$  is a vector element of

$$\mathcal{S}_{N-|\phi|}^\phi := \{z \in \mathbb{Z}_+^L : z_i = 0 \text{ if } \phi_i = 0, \sum_{\{i:\phi_i=1\}} z_i = N - |\phi|\}.$$

Above,  $|\phi|$  is the  $l_1$ -norm of  $\phi$ . Proceeding as before, noting that the constant terms in  $W(x)$  are irrelevant and that no quadratic constraints arise, we need consider only the regions  $\mathcal{X}^\phi$ . Hence, we consider a number of regions whose upper bound is  $2^L$ . Figure 7 provides the bounds obtained via this approach for the network of Example 4.2 with  $N = 20$  as Case A.

Case	Number of Constraints	Number of Variables	Lower Bound	Upper Bound
A	45	18	6.665	8.333
B.1	46	24	6.666	8.333
B.2	49	34	7.046	8.333
B.3	49	40	7.046	8.333
B.4	55	50	7.048	8.333
B.5	56	56	7.232	8.279
B.6	65	66	7.392	8.279
B.7	96	80	7.573	8.130
B.8	105	90	7.579	8.130
B.9	107	96	7.583	8.130
B.10	113	105	7.584	8.130
C.1	17	22	6.622	8.333
C.2	17	22	6.622	8.333
D	24	31	6.663	8.333

Figure 7: Complexity and bounds for the network of Example 4.2.

## 6.2 Two or more quadratics

To tighten the bounds beyond those obtained via consideration of a single quadratic, we allow  $W(x)$  to take different quadratic forms on some of the regions  $\mathcal{X}^\phi$ . We can consider the facets  $X^\phi$  as in Section 6.1 if the possible transitions for all  $x \in \mathcal{X}^\phi$  lead to states whose corresponding quadratic has the same form. Otherwise, we partition the regions for which the next possible states lead to a region with a different quadratic

form into  $\mathcal{Y}^\psi$ .

For the network of Example 4.2, we consider from two to eleven quadratic forms on separate regions  $\mathcal{X}^\phi$  (eleven quadratic forms is equivalent to the approach of Section 4). We first require  $W(x)$  to have the same form for all  $x \in \mathcal{S} - \mathcal{X}^{(0,1,0,0)^T}$  and allow  $W(x)$  to have a separate form for all  $x \in \mathcal{X}^{(0,1,0,0)^T}$ . Thus  $W(x)$  has two pieces. We then allow a third piece for  $W(x)$  to have its own form retaining the unaffected pieces as for the two piece case. Continuing, we increase the number of quadratics, introducing a separate form for each new facet until all facets have separate forms (as in Section 4). Figure 8 provides the new facets allowed to take a separate form. Hence, for Case B.2 of Figure 8,  $W(x)$  takes three forms, one holds for all  $x \in \mathcal{S} - \mathcal{X}^{(0,1,0,0)^T} - \mathcal{X}^{(1,1,0,0)^T}$ , one for all  $x \in \mathcal{X}^{(0,1,0,0)^T}$ , and one for all  $x \in \mathcal{X}^{(1,1,0,0)^T}$ . Figure 7 provides the bounds for each of the cases in Figure 8 for the network of Example 4.2 with  $N = 20$ .

### 6.3 Relaxed simplex approach

Analogously to the arguments presented in Appendix A.2, one may relax the simplex for a given  $\psi$ . That is, one may require a linear inequality to hold on a larger simplex than actually necessary, thus implying the inequality for the smaller simplex. In

Case	Number of Pieces	$\phi$ with New Form
B.1	2	$(0,1,0,0)^T$
B.2	3	$(1,1,0,0)^T$
B.3	4	$(1,0,0,0)^T$
B.4	5	$(0,1,1,0)^T$
B.5	6	$(0,0,1,0)^T$
B.6	7	$(1,0,1,0)^T$
B.7	8	$(1,1,1,0)^T$
B.8	9	$(1,0,0,1)^T$
B.9	10	$(0,0,0,1)^T$
B.10	11	$(0,0,1,1)^T$

Figure 8: Additional facets taking a separate form for the network of Example 4.2.

particular, for a single quadratic, the inequality we seek to verify for a particular  $\psi$  is

$$\begin{aligned}
J \geq \sum_{i \in \mathcal{I}(\phi(\psi))} \mu_i \left[ \frac{1}{L} + r_{i+1} - r_i + N(p_{i+1} - p_i) + x^T Q(e_{i+1} - e_i) \right. \\
\left. + \frac{1}{2}(e_{i+1} - e_i)^T Q(e_{i+1} - e_i) \right],
\end{aligned}$$

for all  $x \in \mathcal{Y}^\psi$ . This inequality will be implied by considering the inequalities on a larger simplex, i.e.,  $x \in \{x \in \mathbb{Z}_+^L : x_j = 0 \text{ if } \psi_j = 0, \sum_{i=1}^L x_i = N\}$ . Equivalently, requiring

$$J \geq \sum_{i \in \mathcal{I}(\phi(\psi))} \mu_i \left[ \frac{1}{L} + r_{i+1} - r_i + N(p_{i+1} - p_i) + N(q_{i+1,j} - q_{i,j}) \right]$$

$$+ \frac{1}{2}q_{i+1,i+1} + \frac{1}{2}q_{i,i} - q_{i,i+1} \Big],$$

for all  $j$  with  $\psi_j \neq 0$  will imply the desired inequality.

Using this relaxation, one can show that the number of inequalities collapses, as many of the inequalities are identical. Combining these observations with a simple two piece quadratic function  $W(x)$  allows us to both improve the bounds slightly and obtain a great measure of control on the complexity of the resulting LPs. Consider  $W(x)$  to have the form  $W''(x)$  on the facet  $\mathcal{X}^{(0,\dots,0,1,0,\dots,0)^T}$ , where the non-zero term in  $\phi = (0, \dots, 0, 1, 0, \dots, 0)^T$  corresponds to the lowest priority buffer at some station (no bound improvement is obtained otherwise through this approach), and the form  $W'(x)$  for the remaining states in the state space. Let  $k$  denote the index of the non-zero element of  $\phi$ . Further, let  $\mathcal{C}$  denote the set of inequalities

$$\begin{aligned} J \geq & \mu_i \left[ \frac{1}{L} + r_{i+1} - r_i + N(p_{i+1} - p_i) + N(q_{i+1,j} - q_{i,j}) \right. \\ & \left. + \frac{1}{2}(e_{i+1} - e_i)^T Q(e_{i+1} - e_i) \right] \\ & + \sum_{\sigma \neq \sigma(j)} \mu_{i_\sigma} \left[ \frac{1}{L} + r_{i_\sigma+1} - r_{i_\sigma} + N(p_{i_\sigma+1} - p_{i_\sigma}) + N(q_{i_\sigma+1,j} - q_{i_\sigma,j}) \right. \\ & \left. + \frac{1}{2}(e_{i_\sigma+1} - e_{i_\sigma})^T Q(e_{i_\sigma+1} - e_{i_\sigma}) \right], \end{aligned}$$

for all  $i \in \sigma(j)$  with  $\theta(i) \leq \theta(j)$ , for all  $i_\sigma \in Buff(\sigma) \cup \Delta$ , and for all  $j \in \{1, 2, \dots, L\}$ .

Above, for  $i_\sigma = \Delta$  we understand that the corresponding term in the inequality is 0.

Now, let  $\mathcal{C}_k$  denote the set of inequalities  $\mathcal{C}$  without the inequality

$$J \geq \mu_k \left[ \frac{1}{L} + r_{k+1} - r_k + N(p_{k+1} - p_k) + N(q_{k+1,k} - q_{k,k}) \right. \\ \left. + \frac{1}{2}(e_{k+1} - e_k)^T Q (e_{k+1} - e_k) \right].$$

An upper bound LP is then given by *Min*  $J$ , subject to the inequalities  $\mathcal{C}_k$  and the inequalities obtained from the regions  $\mathcal{Y}^{2e_k}$  and  $\mathcal{Y}^{2e_k+e_{k-1}}$  (via the approach of Section 4).

For the network of Example 4.2, the above LP may be reduced further to account for the presence of the virtual station. In particular, any constraint of  $\mathcal{C}$  for which  $i = 2$  and  $i_\sigma = 4$  or  $i = 4$  and  $i_\sigma = 2$  may be eliminated from consideration. Figure 7 gives the bounds thereby obtained as Case C.1 for  $k = 1$  and Case C.2 for  $k = 3$  for the network of Example 4.2 with  $N = 20$ .

Similarly, one may develop an LP bound by allowing  $W''(x)$  to be the quadratic form on the facets  $\mathcal{X}^{e_k}, \mathcal{X}^{e_l}, \mathcal{X}^{e_k+e_l}$  and  $W'(x)$  the quadratic form elsewhere. In this case, the constraints  $\mathcal{C}_{k,l}$  are those of  $\mathcal{C}$  for  $j \neq k, j \neq l$ . An LP bound is obtained by requiring the inequalities  $\mathcal{C}_{k,l}$  together with the constraints obtained by consideration of the regions  $\mathcal{Y}^\psi$  whose inequalities are not ensured by the inequalities of  $\mathcal{C}_{k,l}$ . The bounds for  $k = 1, l = 2$  for the network of Example 4.2 is given as Case D of Figure 7.

For comparison, the LPs of [9] require 19 variables and 17 constraints.



## 7 Concluding Remarks

We have shown how to construct linear programming bounds for the throughput of closed reentrant lines. The approach arises from the examination of the average cost inequality and a recognition that for certain scheduling policies (namely, buffer priority policies and the class of all non-idling scheduling policies) the feasible next state transitions are invariant within certain regions of the state space. Proposing a piecewise quadratic function as a surrogate for the differential cost function, with a constant form on each transition invariant region, allows one to obtain quadratic inequalities. If these inequalities can be satisfied, then a performance bound may be obtained.

For a fixed customer population  $N$ , we show that one may extract the quadratic terms in the inequalities, and apply a simplex equivalence condition for the linear plus constant terms in the inequalities, to obtain a linear program to bound the throughput  $\lambda_N$  for fixed  $N$ . Further, proposing a functional form for the surrogate of the differential cost function allows us to obtain linear programs for limiting infinite population throughput bounds as well as functional throughput bounds parameterized by  $N$ . In all cases, the bounds obtained are provably tighter than those previously known, though there is a computational price to be paid as the state space must be partitioned into a number of regions whose upper bound is  $3^L$ . However, one can

restrict the complexity, again retaining superiority over earlier bounds, by entertaining only a limited number of  $\phi$ 's on which the quadratics are allowed to differ and relaxing the simplex conditions to a simplex of larger size.

## 8 Appendix A.1: The definitions of $k^\psi$ , $s^\psi$ and $M^\psi$ .

$$\begin{aligned}
k^\psi := & \sum_{i \in \mathcal{I}(\phi(\psi))} \mu_i \left[ \frac{1}{L} + c^{\phi(\psi - e_i + e_{i+1})} - c^{\phi(\psi)} \right. \\
& + (d^{\phi(\psi - e_i + e_{i+1})} - d^{\phi(\psi)})(e^T \psi) \\
& + (f^{\phi(\psi - e_i + e_{i+1})} - f^{\phi(\psi)})(e^T \psi)^2 \\
& + r_{i+1}^{\phi(\psi - e_i + e_{i+1})} - r_i^{\phi(\psi - e_i + e_{i+1})} \\
& + (r^{\phi(\psi - e_i + e_{i+1})} - r^{\phi(\psi)})^T \psi \\
& + (p_{i+1}^{\phi(\psi - e_i + e_{i+1})} - p_i^{\phi(\psi - e_i + e_{i+1})})(e^T \psi) \\
& + (p^{\phi(\psi - e_i + e_{i+1})} - p^{\phi(\psi)})^T \psi (e^T \psi) \\
& + \frac{1}{2}(\psi - e_i + e_{i+1})^T Q^{\phi(\psi - e_i + e_{i+1})}(\psi - e_i + e_{i+1}) \\
& \left. - \frac{1}{2}\psi^T Q^{\phi(\psi)}\psi \right],
\end{aligned}$$

$$\begin{aligned}
s^\psi := & \sum_{i \in \mathcal{I}(\phi(\psi))} \mu_i \left[ (d^{\phi(\psi - e_i + e_{i+1})} - d^{\phi(\psi)})e + 2(f^{\phi(\psi - e_i + e_{i+1})} - f^{\phi(\psi)})e(e^T \psi) \right. \\
& + (r^{\phi(\psi - e_i + e_{i+1})} - r^{\phi(\psi)}) \\
& + (p_{i+1}^{\phi(\psi - e_i + e_{i+1})} - p_i^{\phi(\psi - e_i + e_{i+1})})e \\
& \left. + (p^{\phi(\psi - e_i + e_{i+1})} - p^{\phi(\psi)})(e^T \psi) \right]
\end{aligned}$$

$$\begin{aligned}
& + e(p^{\phi(\psi-e_i+e_{i+1})} - p^{\phi(\psi)})^T \psi \\
& + Q^{\phi(\psi-e_i+e_{i+1})}(\psi - e_i + e_{i+1}) - Q^{\phi(\psi)}\psi,
\end{aligned}$$

$$\begin{aligned}
M^\psi := \sum_{i \in \mathcal{I}(\phi(\psi))} \mu_i & \left[ 2(f^{\phi(\psi-e_i+e_{i+1})} - f^{\phi(\psi)})ee^T + e(p^{\phi(\psi-e_i+e_{i+1})} - p^{\phi(\psi)})^T \right. \\
& + (p^{\phi(\psi-e_i+e_{i+1})} - p^{\phi(\psi)})e^T \\
& \left. + (Q^{\phi(\psi-e_i+e_{i+1})} - Q^{\phi(\psi)}) \right].
\end{aligned}$$

## 9 Appendix A.2.

We prove the superiority of the bounds obtained here for a fixed customer population  $N$  over those of [5] by consideration of a single quadratic function

$$W(x) := (c + Nd + N^2 f) + r^T x + (e^T x)p^T x + \frac{1}{2}x^T Qx, \forall x \in \mathcal{S}.$$

The inequalities of (5,6) thus reduce to

$$\begin{aligned}
J \geq \sum_{i \in \mathcal{I}(\phi(\psi))} \mu_i & \left[ \frac{1}{L} + r_{i+1} - r_i + (N - |\psi|)(p_{i+1} - p_i) \right. \\
& + (e^T \psi)(p_{i+1} - p_i) + (N - |\psi|)(q_{i+1,j} - q_{i,j}) \\
& \left. + \frac{1}{2}(e_{i+1} - e_i)^T Q(e_{i+1} - e_i) + \psi^T Q(e_{i+1} - e_i) \right], \tag{10}
\end{aligned}$$

for all  $\psi$ , for all  $j$  with  $\psi_j = 2$ . It is straightforward to show that these are equivalent to

$$\begin{aligned}
J \geq \sum_{i \in \mathcal{I}(\phi(\psi))} \mu_i & \left[ \frac{1}{L} + r_{i+1} - r_i + N(p_{i+1} - p_i) + x^T Q(e_{i+1} - e_i) \right. \\
& \left. + \frac{1}{2}(e_{i+1} - e_i)^T Q(e_{i+1} - e_i) \right], \tag{11}
\end{aligned}$$

for all  $x \in \mathcal{Y}^\psi$ , for all  $\psi$ . These are implied by the same inequalities on a larger simplex, i.e., requiring for each  $\psi$  that the inequality hold for  $\forall x \in \{x \in \mathcal{S} : x_i = 0 \text{ if } \psi_i = 0, e^T x = N\}$ . This is equivalent to

$$J \geq \sum_{i \in \mathcal{I}(\phi(\psi))} \mu_i \left[ \frac{1}{L} + r_{i+1} - r_i + N(p_{i+1} - p_i) + N(q_{i+1,j} - q_{i,j}) + \frac{1}{2}q_{i+1,i+1} + \frac{1}{2}q_{i,i} - q_{i,i+1} \right], \quad (12)$$

for all  $\psi$ , for all  $j$  with  $\psi_j \neq 0$ .

These constraints along with the cost function  $\text{Min } J$ , are equivalent to the dual of the LP of [5] for an upper bound. Hence, any feasible solution to this dual of an upper bound from [5], i.e., any solution satisfying the constraints of (12), implies (11) which is equivalent to (11), proving the superiority.

The superiority of the lower bound is proved similarly.

## 10 Appendix A.3: The definitions of $l^\psi$ , $t^\psi$ , and $R^\psi$ .

$$\begin{aligned} \ell^\psi := & \sum_{i \in \mathcal{I}(\phi(\psi))} \mu_i \left[ \frac{\psi^T e}{L} + c^{\phi(\psi - e_i + e_{i+1})} - c^{\phi(\psi)} \right. \\ & + (d^{\phi(\psi - e_i + e_{i+1})} - d^{\phi(\psi)})(e^T \psi) \\ & + (f^{\phi(\psi - e_i + e_{i+1})} - f^{\phi(\psi)})\psi^T e e^T \psi \\ & + (r_{i+1}^{\phi(\psi - e_i + e_{i+1})} - r_i^{\phi(\psi - e_i + e_{i+1})}) + (r^{\phi(\psi - e_i + e_{i+1})} - r^{\phi(\psi)})^T \psi \\ & \left. + \psi^T e p^{\phi(\psi - e_i + e_{i+1})^T} (\psi - e_i + e_{i+1}) - \psi^T e p^{\phi(\psi)^T} \psi \right] \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2}(\psi - e_i + e_{i+1})^T Q^{\phi(\psi - e_i + e_{i+1})} (\psi - e_i + e_{i+1}) \\
& - \frac{1}{2}\psi^T Q^{\phi(\psi)} \psi \Big],
\end{aligned}$$

$$\begin{aligned}
t^\psi := \sum_{i \in \mathcal{I}(\phi(\psi))} \mu_i \Big[ & \frac{e}{L} + (d^{\phi(\psi - e_i + e_{i+1})} - d^{\phi(\psi)})e + 2(f^{\phi(\psi - e_i + e_{i+1})} - f^{\phi(\psi)})e e^T \psi \\
& + (r^{\phi(\psi - e_i + e_{i+1})} - r^{\phi(\psi)}) \\
& + (p_{i+1}^{\phi(\psi - e_i + e_{i+1})} - p_i^{\phi(\psi - e_i + e_{i+1})})e \\
& + (p^{\phi(\psi - e_i + e_{i+1})} - p^{\phi(\psi)})(e^T \psi) \\
& + e(p^{\phi(\psi - e_i + e_{i+1})} - p^{\phi(\psi)})^T \psi \\
& + Q^{\phi(\psi - e_i + e_{i+1})}(\psi - e_i + e_{i+1}) \\
& - Q^{\phi(\psi)} \psi \Big],
\end{aligned}$$

$$\begin{aligned}
R^\psi := \sum_{i \in \mathcal{I}(\phi(\psi))} \mu_i \Big[ & 2(f^{\phi(\psi - e_i + e_{i+1})} - f^{\phi(\psi)})e e^T + e(p^{\phi(\psi - e_i + e_{i+1})} - p^{\phi(\psi)})^T \\
& + (p^{\phi(\psi - e_i + e_{i+1})} - p^{\phi(\psi)})e^T \\
& + (Q^{\phi(\psi - e_i + e_{i+1})} - Q^{\phi(\psi)}) \Big].
\end{aligned}$$

## 11 Appendix A.4.

We consider

$$W(x) = \left[ \frac{c + Nd + N^2 f + r^T x + Np^T x + \frac{1}{2}x^T Qx}{N + \nu} \right], \forall x \in \mathcal{S},$$

so that the inequalities of (7,8,9) become

$$\begin{aligned}\alpha' &\geq \sum_{i \in \mathcal{I}(\phi(\psi))} \mu_i \left[ \frac{1}{L} + p_{i+1} - p_i + q_{i+1,j} - q_{i,j} \right], \forall \psi, \forall j \text{ with } \psi_j = 2. \\ 0 &\geq \sum_{i \in \mathcal{I}(\phi(\psi))} \mu_i \left[ \frac{\nu}{L} + r_{i+1} - r_i + \frac{1}{2}q_{i+1,i+1} + \frac{1}{2}q_{i,i} - q_{i,i+1} \right. \\ &\quad \left. - (q_{i+1,j} - q_{i,j})(\psi^T e) + \psi^T Q(e_{i+1} - e_i) \right],\end{aligned}$$

for all  $\psi$ , for all  $j$  with  $\psi_j = 2$ . This is equivalent to

$$\begin{aligned}\alpha' N &\geq \sum_{i \in \mathcal{I}(\phi(\psi))} \mu_i \left[ \frac{N + \nu}{L} + r_{i+1} - r_i + N(p_{i+1} - p_i) + \frac{1}{2}q_{i+1,i+1} \right. \\ &\quad \left. + \frac{1}{2}q_{i,i} - q_{i,i+1} + N(q_{i+1,j} - q_{i,j}) \right. \\ &\quad \left. - |\psi|(q_{i+1,j} - q_{i,j}) + \psi^T Q(e_{i+1} - e_i) \right],\end{aligned}$$

for all  $N \geq 0$ , for all  $\psi$ , for all  $j$  with  $\psi_j = 2$ , which is in turn the same as

$$\begin{aligned}\alpha' N &\geq \sum_{i \in \mathcal{I}(\phi(\psi))} \mu_i \left[ \frac{N + \nu}{L} + r_{i+1} - r_i + N(p_{i+1} - p_i) \right. \\ &\quad \left. + \frac{1}{2}q_{i+1,i+1} + \frac{1}{2}q_{i,i} - q_{i,i+1} + N(q_{i+1,j} - q_{i,j}) \right. \\ &\quad \left. + (z + \psi)^T Q(e_{i+1} - e_i) \right],\end{aligned}$$

for all  $N \geq 0$ , for all  $\psi$ , and for all  $z \in \mathcal{S}_{N-|\psi|}^\psi$ . This however is implied by a relaxation of the simplex (as in Appendix A.2), so that

$$\begin{aligned}\alpha' N &\geq \sum_{i \in \mathcal{I}(\phi(\psi))} \mu_i \left[ \frac{N + \nu}{L} + r_{i+1} - r_i + N(p_{i+1} - p_i) \right. \\ &\quad \left. + \frac{1}{2}q_{i+1,i+1} + \frac{1}{2}q_{i,i} - q_{i,i+1} + N(q_{i+1,j} - q_{i,j}) \right],\end{aligned}$$

for all  $N \geq 0$ , for all  $\psi$ , and for all  $j$  with  $\psi_j \neq 0$ , implies the previous inequalities.

Now, this last is equivalent to

$$\alpha' \geq \sum_{i \in \mathcal{I}(\phi(\psi))} \mu_i \left[ \frac{1}{L} + p_{i+1} - p_i + q_{i+1,j} - q_{i,j} \right], \quad (13)$$

$$0 \geq \sum_{i \in \mathcal{I}(\phi(\psi))} \mu_i \left[ \frac{\nu}{L} + r_{i+1} - r_i + \frac{1}{2}q_{i+1,i+1} + \frac{1}{2}q_{i,i} - q_{i,i+1} \right], \quad (14)$$

for all  $\psi$ , for all  $j$  with  $\psi_j = 2$ . The inequalities of (13) are a dual of the dual of the feasibility conditions of [4], where  $\alpha$  is the value of their limiting infinite population LP, for the functional bounds. Take any  $p, Q$  satisfying them. Then,

$$\begin{aligned} r_1 &= 0 \\ r_i &= \frac{-(i-1)}{L} \left[ \sum_{j=1}^L (q_{i+1,j} - q_{i,j}) \right] - \frac{1}{2}q_{i,i} - \frac{1}{2}q_{11} - \sum_{j=2}^{i-1} q_{jj} + \sum_{j=1}^{i-1} q_{j,j+1} \end{aligned}$$

is feasible for (14), with

$$\nu = \sum_{j=1}^L (q_{j,j+1} - q_{j,j}).$$

This is the objective function of [4], proving the superiority.

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