Capacity of Wireless Networks: Protocol and Physical Models

P. R. Kumar

Dept. of Electrical and Computer Engineering, and Coordinated Science Lab
University of Illinois, Urbana-Champaign

Email: prkumar@illinois.edu
Web: http://decision.csl.illinois.edu/~prkumar
How much traffic can wireless networks carry?

(Or what is the capacity of wireless networks?)
And how should information be transferred in wireless networks?
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Models of Wireless Networks
Multi-hop Wireless networks

- Communication networks formed by nodes with radios
  - Spontaneously deployable anywhere
  - Automatically adaptive to number of nodes, traffic requirements, locations

- “Multi-hop transport”
  - Nodes relay packets until they reach their destinations
Two fundamental properties of the wireless medium

- It is subject to fading and attenuation
  - Signals get distorted
  - Time varying channel
  - Unreliable

- It is a shared medium
  - Users share the same spectrum
  - Users are located next to each other
  - Transmissions can interfere with each other
  - So users need to cooperate to use the medium
Spatial reuse of spectrum

- Spatial reuse of frequency in cellular systems
Shared nature of wireless medium

- Packets can “collide” destructively
  - Destructive interference
  - Nothing can be decoded from two concurrent transmissions in same region
One model for successful sharing

- Receiver not in vicinity of an interfering transmission

\[ r_2 = (1 + \Delta) r_2 \]

\[ (1 + \Delta) r_1 \]
Spatial reuse of spectrum

- Spatial reuse of frequency in cellular systems
Other models for successful sharing

- Signal to Noise Ratio (SNR)
  \[
  \text{Signal to Noise Ratio} = \text{SNR} := \frac{\text{Received Signal Strength}}{\text{Noise}} = \frac{P_i}{r_i^\alpha}
  \]

- Signal to \textit{Interference plus Noise} Ratio (SINR)
  \[
  \text{SINR} := \frac{\text{Received Signal Strength}}{\text{Interference Strength} + \text{Noise}} = \frac{P_i}{r_i^\alpha} \geq \sum_{j \neq i} \frac{P_j}{r_j^\alpha} + N
  \]

- Model 2: Reception successful if SINR exceeds a threshold:
  \[
  \text{SINR} = \frac{P_i}{r_i^\alpha} \geq \beta \geq \sum_{j \neq i} \frac{P_j}{r_j^\alpha} + N
  \]

- Model 3: Transmitter-to-Receiver Communication Rate depends on SINR:
  \[
  \text{Rate} = B \log \left( 1 + \frac{P_i r_i^{-\alpha}}{N + \sum_{j \neq i} P_j r_j^{-\alpha}} \right) \text{ bps}
  \]
A framework for studying wireless networks

- **Model**
  - Disk of area $A$ sq.m
  - $n$ nodes
  - Each can transmit at $W$ bits/sec

- **Wireless channel is a shared medium**
  - Packets are successfully received when there is no local interference

- **How much information can such wireless networks carry?**
  - **Throughput for each node**: Measured in Bits/Sec
  - **Transport capacity of entire network**: Measured in Bit-Meters/Sec
  - **Scaling with the number of nodes** $n$
Model for successful decoding of packet

- **Protocol Model**

Receiver $R$ should be

(i) within range $r$ of its own transmitter $T$

(ii) outside footprint $(1+\Delta)r'$ of any other transmitter $T'$ using range $r'$
Best Case Analysis of Protocol Model
Transmissions consume area

\[ A \text{ sq.m} \]

\( n \) nodes
Transmissions consume area

\[ A \text{ sq.m} \]

\[ (1+\Delta)r_1 \]

\[ (1+\Delta)r_2 \]

\[ n \text{ nodes} \]
Transmissions consume area

\[ r_1 \]

\[ (1+\Delta)r_1 \]

\[ r_2 \]

\[ (1+\Delta)r_2 \]
Transmissions consume area

\[ r_1 (1 + \Delta) r_1 \]
Transmissions consume area

\[ r_1 \geq (1 + \Delta) r_1 \]
Transmissions consume area

\( r_1 \geq (1 + \Delta) r_1 \geq \Delta r_1 \)
Transmissions consume area $\geq \Delta r_1$
Transmissions consume area

$\text{transmissions consume area}$

\[(1+\Delta)r_1\]

\[(1+\Delta)r_2\]
Transmissions consume area

\[ r^2 (1+\Delta) r^2 \]
Transmissions consume area

\[ r^2 \geq (1+\Delta)r_2 \]
Transmissions consume area

\[ r^2 \geq (1 + \Delta)r^2 \geq \Delta r^2 \]
Transmissions consume area

$\geq \Delta r_2$
Transmissions consume area ≥ Δr₁
Transmissions consume area

\[ \Delta r_1/2 \geq \Delta (r_1 + r_2)/2 \geq \Delta r_2/2 \]
Transmissions consume area

\[ \Delta r_1/2 \]

\[ r_1 \]

\[ \Delta r_2/2 \]

\[ r_2 \]
Transmissions consume area \[ \Delta r_1/2 \]
Transmissions consume area

\[ \Delta r_1 / 2 \]

\[ n \text{ nodes} \]

\[ A \text{ sq.m} \]
Transmissions consume area

\[ A \text{ sq.m} \]

\[ \Delta r_1/2 \]

\[ \Delta r_2/2 \]

\[ r_1 \]

\[ r_2 \]

\[ n \text{ nodes} \]
Transmissions consume area

\[ A \text{ sq.m} \]

\[ \Delta r_1/2 \quad \Delta r_2/2 \quad \Delta r_3/2 \quad \Delta r_4/2 \quad \Delta r_5/2 \quad \Delta r_6/2 \]

But Total Area = \( A \)

So \[
\sum_{i=1}^{n/2} \frac{\pi \Delta^2 r_i^2}{16} \leq A
\]
The area constraint

\[ \sum_{i=1}^{n/2} \frac{\pi \Delta^2 r_i^2}{16} \leq A \]

\[ \sum_{i=1}^{n/2} r_i^2 \leq \frac{16A}{\pi \Delta^2} \]

\[ \frac{1}{n/2} \sum_{i=1}^{n/2} r_i^2 \leq \frac{32A}{\pi \Delta^2} \cdot \frac{1}{n} \]
Convexity

- A convex function: \( f(r) = r^2 \)

Square of Average \( \leq \) Average of squares

Eg. \( \left( \frac{3 + 5}{2} \right)^2 \leq \frac{3^2 + 5^2}{2} \) (or \( 16 \leq 17 \))

So

\[
\left( \frac{1}{n/2} \sum_{i=1}^{n/2} r_i \right)^2 \leq \frac{1}{n/2} \sum_{i=1}^{n/2} r_i^2 \leq \frac{32A}{\pi \Delta^2} \cdot \frac{1}{n}
\]
Convexity

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So

\[
\left( \frac{1}{n/2} \sum_{i=1}^{n/2} r_i \right)^2 \leq \frac{1}{n/2} \sum_{i=1}^{n/2} r_i^2 \leq \frac{32A}{\pi\Delta^2} \cdot \frac{1}{n}
\]

Hence

\[
\sum_{i=1}^{n/2} r_i \leq \sqrt{\frac{8A}{\pi\Delta^2} \cdot n}
\]
Bound on Transport capacity: Bit-meters/second pumped by network

- Total bit-meters/second pumped by the network is \( \sum_{i=1}^{n/2} W r_i \)

- Remember \( \sum_{i=1}^{n/2} r_i \leq \sqrt{\frac{8A}{\pi \Delta^2}} \cdot n \)

- So \( \sum_{i=1}^{n/2} W r_i \leq W \sqrt{\frac{8A}{\pi \Delta^2}} \cdot n \)

- Hence

Bit-meters pumped by the network \( \leq W \sqrt{\frac{8}{\pi \Delta^2}} \cdot A \cdot n \)
Feasibility of $\Theta(W \sqrt{An})$ bit-meters/sec

- $n/2$ nearest neighbor connections
- Each of distance $r = \sqrt{2A/n}$
- Total bit meters of entire network

$$(n/2)Wr = \frac{1}{\sqrt{8}} W \sqrt{An}$$

- Basically the wireless network provides every user a throughput of $W$ bps to its nearest neighbor
Scaling law for capacity

- How much information can wireless networks transfer with this mode of operation?
  - Transport capacity is $\Theta(W\sqrt{An})$ bit-meters/second
    - Aggregate pumping capacity of the network (Gupta & K ’00)
  - Implications of square-root law
    - If equitably divided, each node can send $\Theta\left(W\sqrt{\frac{A}{n}}\right)$ bit-meters/sec
    - Law of diminishing returns in this scaling
Best possible scenario

- **Optimal network**
  - Optimally located nodes, destinations, demands for OD-pairs
  - Optimal spatial and temporal scheduling, routes, ranges for each transmission

- **Protocol Model:** Network can transport \( \Theta(W\sqrt{An}) \) bit-meters/sec

\[
\frac{W}{1+2\Delta} \frac{n}{\sqrt{n+\sqrt{8\pi}}} \leq \text{Best case capacity} \leq \sqrt{\frac{8W}{\pi\Delta}} \sqrt{n} \quad \text{bit-meters/sec}
\]

  for Protocol Model

- If equitably divided, each node can send \( \Theta\left(W\sqrt{\frac{A}{n}}\right) \) bit-meters/sec
Another intuitive way of understanding result

- Best Range: Tradeoff between short and long hops
  - If range is $r$, distance traveled on each hop is $r$
  - So number of hops to travel one unit of distance $\approx \frac{1}{r}$
  - Space used by each transmission = $r^2$
  - Total Space used $\approx \frac{1}{r} \cdot r^2 = r$

Or Distance-Rate product $\approx r \times \frac{1}{r^2} = \frac{1}{r}$

- So best to use smallest range $r$

- However smallest range that keeps network connected = $\frac{1}{\sqrt{n}}$
Same result also holds if channel is split into several sub-channels

- Domain is a disk of unit area
- There are $n$ nodes in the domain
- Each node can transmit at $W$ bits/sec
- Channel can be split into $M$ sub-channels of capacities $W_1, W_2, \ldots, W_M$ bits/sec with $\sum_{m=1}^{M} W_m = W$
- Assume slots of length $\tau$
  - $W_m \tau$ bits can be transmitted in slot $s$ in subchannel $m$
Sharpest results for Best Case of Protocol Model
Sharpest results for Best Case: Protocol Model

- **Protocol Model** (Agarwal & K ’03)

\[
\sqrt{\frac{1}{\pi}} \frac{W}{\sqrt{(1+\Delta)\Delta \sqrt{2+\Delta}}} \cdot \sqrt{A} \cdot \sqrt{n} \leq \text{Transport capacity} \leq \sqrt{\frac{8}{\pi}} \frac{W}{\sqrt{(1+\Delta)\Delta \sqrt{2+\Delta}}} \cdot \sqrt{A} \cdot \sqrt{n}
\]

- Upper and lower bounds differ by only a factor of \(\sqrt{8}\)
  - A sharper characterization of “exclusion region”
  - Study of general antenna patterns, directional antennas, etc
Best Case Analysis of Physical Model
The Physical SINR Model

- Physical Model: Signal-to-Interference-Plus Noise Ratio (SIR) Model

\[
\text{SINR Ratio} = \frac{P_i r_i^{-\alpha}}{N + \sum_{j \neq i} P_j r_j^{-\alpha}} \geq \beta
\]

- \( P_i \) = power of \( i \)-th node
- \( N \) = Noise power
- \( r_j \) = Distance of \( j \)-th transmitter from given receiver
- \( r^{-\alpha} \): Signal Power Path Loss, \( \alpha > 2 \)
- \( \beta \) = SIR for successful reception

- Will show a simple derivation of a \( O(n^{(\alpha-1)/\alpha}) \) bound (Gupta & K ‘04)
$O(n^{\alpha-1/\alpha})$ bound: Physical Model

**Idea**

SINR constraint also implies a space constraint:

Recall SINR requirement:

$$\frac{P_i r_i^{-\alpha}}{N + \sum_{j \neq i} P_j r_j^{-\alpha}} \geq \beta$$

Including signal power in denominator:

$$\frac{P_i r_i^{-\alpha}}{N + \sum_j P_j r_j^{-\alpha}} \geq \frac{\beta}{\beta+1}$$

So

$$r_i^{\alpha} \leq \frac{(\beta + 1) P_i}{\beta N + \beta \sum_j P_j r_j^{-\alpha}}$$
SIR requirement consumes space

- So \( r_i^\alpha \leq \frac{(\beta+1)P_i}{\beta N + \beta \sum_j P_j r_j^{-\alpha}} \)

- However \( r_j \leq \frac{2}{\sqrt{\pi}} \) (since area of disk = 1)

- Hence \( r_i^\alpha \leq \frac{(\beta+1)P_i}{\beta N + \beta \left(\frac{\pi}{4}\right)^\alpha \sum_j P_j} \)

- So \( \sum_{R_i} R_i^\alpha \leq \frac{(\beta+1)}{\beta \left(\frac{\pi}{4}\right)^\alpha \sum_j P_j} \)

- Now use convexity of \( R^\alpha \) rather than that of \( r^2 \)
Sharpest results for Best Case of Physical Models
Sharpest results for Best Case: Physical Models

- **Physical Model** (Agarwal and K ’04)
  - Transport capacity is $\Theta(\sqrt{n})$

- **Generalized Physical Model** (Agarwal and K ’04)
  - Adaptive coding is used so that bit-rate depends on SINR

$$\text{Rate} = B \log \left( 1 + \frac{P_{r_i}^\delta}{N + \sum_{j \neq i} P_{r_j}^\delta} \right) \text{ bps}$$

  (Shannon’s formula for capacity of a channel with additive white Gaussian noise)

  - Transport capacity is still $\Theta(\sqrt{n})$

  (Agarwal and K ’04)
Analysis of The Random Case
Scaling Law for Random Networks

- \( n \) nodes randomly located in disk of unit area
  - Each node chooses random destination
  - Equal throughput \( \lambda \) bits/sec for all OD pairs
  - Each node chooses the same range \( r \)

- Each node can send \( \Theta\left(\frac{1}{\sqrt{n\log n}}\right) \) bits/sec even with
  - With best choice of spatio-temporal scheduling, ranges and routes

- **Theorem**: Throughput that can be supported (Gupta & K ’00)
  \[
  \lim_{n \to \infty} \Pr\left(\frac{c}{\sqrt{n\log n}} \text{ is feasible}\right) = 1, \quad \text{and}
  \lim_{n \to \infty} \Pr\left(\frac{c'}{\sqrt{n\log n}} \text{ is feasible}\right) = 0
  \]

(i) Random case = Nearly best case
(ii) It is nearly optimal to use common range for all nodes

Sharp cutoff phenomenon
Model of randomly formed networks

- Domain = Surface of sphere of unit area
  - No edge effects (can generalize to plane)
- \( n \) identical nodes randomly located (uniform, iid)
- Each node can transmit at \( W \) bits/sec
- Range of each node’s transmission is \( r(n) \)
- Destination nodes randomly chosen
  - Node closest to a randomly chosen (uniform, iid) point
- Each node sends \( \lambda(n) \) bits/sec to its destination

**Theorem** The capacity of a randomly formed network is \( \Theta\left(\frac{1}{\sqrt{n \log n}}\right) \) bits/sec
Constructive proof of capacity

- **Voronoi Tessellation**
  - Generators $a_1, a_2, \ldots, a_p$
  - Cell $V(a_i) =$ region closest to $a_i$

- Choose a tessellation so that
  - Every cell contains a disk of radius $\rho$
  - Every cell is contained in a disk of radius $2\rho$
    - Procedure: Add a generator $2\rho$ away from other generators
    - Stop when no more possible
  - Cells neither too fat nor too thin

- Choose $\rho(n)$ so Area of disk ($\approx \pi \rho^2(n)$) = $\frac{100 \log n}{n}$
Choose transmission range \( r(n) = 8\rho(n) \)

- Neighboring cells can communicate
  - Each cell is contained in disk of radius \( 2\rho(n) \)
  - Range is \( 8\rho(n) \)
Interfering neighbors

- **Definition**  
  Interfering Neighbors
  
  - Cell $c$ and Cell $c'$ are interfering neighbors if
    
    - Some point in one cell is within a distance $(2+\Delta) r(n)$ of some point in other cell

- **Transmissions**
  
  Transmissions from cell $c$ and $c'$ can never collide if they are not interfering neighbors
Number of interfering neighbors

- Every cell has no more than $c_1$ interfering neighbors
A transmission schedule for cells

- There is a transmission schedule such that every cell gets one transmission slot in every \((1+c_1)\) slots

  - Proof based on graph coloring
    - Draw an edge between two cells if they are interfering neighbors
    - Degree of graph \(\leq c_1\)
    - Can vertex color the graph with no more than \((1+c_1)\) colors
    - All nodes of same color transmit in a slot

  - Proof for SIR model based on separation distances of other receivers
**OD Pairs, Traffic and “Lines”**

- **Nodes** Randomly located \( \{X_1, X_2, \ldots, X_n\} \)

- **Destinations**
  - \( Y_i \) randomly located point (uniform iid)
  - \( X_{\text{dest}(i)} = \text{nearest node to } Y_i \)

- **OD Pair** = \((X_i, X_{\text{dest}(i)})\)

- **Traffic of OD Pair** = \( \lambda \)

- **\( L_i \)** = line joining \( X_i \) and \( Y_i \)

- \( \{L_i\} \) are iid
Routes

- Packets hop from one cell intersecting $L_i$ to next cell intersecting $L_i$
When is scheme feasible?

Scheme is feasible if

- Every cell contains at least one node to act as a relay
- Each cell can handle all the traffic passing through it
How many “lines” can a cell handle?

- Each cell gets one slot out of every \((1 + c_1)\) slots.
- When it transmits it can transmit at \(W\) bits/sec.
- Hence each cell can carry \(\frac{W}{1 + c_1}\) bits/sec.
- Each line \(L_i\) passing through cell consumes \(\lambda\) bits/sec.
- Hence each cell can handle at most \(\frac{W}{\lambda(1 + c_1)}\) lines.
To show feasibility of scheme ....

We need to show

- **Every** cell has at least one node

- **Every** cell has less than \( \frac{W}{\lambda(1 + c_1)} \) lines passing through it
Every cell contains a node

- Every cell contains a disk of probability \( (100 \log n)/n \)
- So \( \Pr(\text{Given cell contains no nodes}) \leq (1 - \frac{100 \log n}{n})^n \)

Number of cells \( \leq \frac{n}{100 \log n} \)

\[ \Pr(\text{Every cell contains a node}) \geq 1 - \frac{n}{100 \log n} \left(1 - \frac{100 \log n}{n}\right)^n \]

\[ \lim_{n \to \infty} \Pr(\text{Every cell contains a node}) = 1 \]

- Every cell contains a node with probability approaching one as \( n \to \infty \) (whp)
Weak Law of large numbers

- \{X_1, X_2, \ldots, X_n\} are iid, common distribution \( P \)

- When \( n \) is large enough

\[
\Pr \left( \left| \frac{\text{# of points in } G}{n} - P(G) \right| > \varepsilon \right) < \delta
\]
Shattering

- $\mathcal{S}$ = set of subsets

- $B = \{z_1, z_2, \ldots, z_p\}$ = a finite set of $p$ points

- **$B$ is shattered by $\mathcal{S}$** if
  - For every subset $A$ of $B$ there is a $G \in \mathcal{S}$ with $G \cap B = A$

- **Example**  $\mathcal{S}$ = set of all rectangles
  $B = \{z_1, z_2, z_3, z_4\}$
Vapnik-Chervonenkis dimension

- Vapnik-Chervonenkis Dimension \( VC(\mathcal{S}) \)
  - Size of largest set than can be shattered by \( \mathcal{S} \)

Example

\( \mathcal{S} = \) set of all rectangles

There is no set with 5 points that can be shattered

\( VC(\mathcal{S}) = 4 \)
Uniform law of large numbers

- Suppose $VC(\mathcal{F}) = d$

- When

$$\Pr\left( \sup_{G \in \mathcal{F}} \left| \frac{\# \text{ of points in } G}{n} - P(G) \right| > \varepsilon \right) \leq 4 \left( (2n)^{2d+1} + 1 \right) e^{-\frac{-\varepsilon^2 n}{8}}$$

Uniformity over $\mathcal{F}$

- When $n > n(\varepsilon, \delta, d)$, $\Pr\left( \sup_{G \in \mathcal{F}} \left| \frac{\# \text{ of points in } G}{n} - P(G) \right| > \varepsilon \right) \leq \delta$

- Every set in $\mathcal{F}$ has nearly the mean number of points
VC dimension of disks on the plane

- **Theorem**  \( \text{VC(set of disks in the plane)} = 3 \)

- **Proof**

  Suppose \( \{x_1, x_2, x_3, x_4\} \) can be shattered

  Suppose \( \angle x_1 + \angle x_3 \geq 180^\circ \)

  Then \( \angle x_1 < \angle a \) and \( \angle x_3 < \angle c \)

  But \( \angle a + \angle c = 180^\circ \)

  So \( \angle x_1 + \angle x_3 < 180^\circ \)
From $S^2$ to $R^2$ and back: The inversion map

- Let $f(z) = \frac{z}{\|z\|^2}$ the inversion map

- Properties:

  $f : $ Punctured sphere $\rightarrow$ Plane
  $f : $ Disks on $S^2$ $\rightarrow$ Disks on $R^2$ Plane

  $f^{-1}(z) = f(z)$

- Theorem

  $VC($disks smaller than hemisphere on $S^2$) = 3

  Proof If 4 points shattered, rotate to lower hemisphere
Probability of a line passing through a cell

- Each cell is contained in a disk of area \( \frac{(400 \log n)}{n} \)

- Radius of disk \( \gamma(n) \leq \sqrt{\frac{400 \log n}{n}} \)

\[
\Pr(\text{Line } L \text{ intersects cell}) \leq \int_0^c \min \left[ c, \left( \frac{c}{x} \sqrt{\frac{\log n}{n}} \right) \right] c \left( x + \sqrt{\frac{400 \log n}{n}} \right) dx
\]

\[
\leq \int_0^c \left( \frac{c}{x} \sqrt{\frac{\log n}{n}} \right) x dx + \int_0^c c \sqrt{\frac{400 \log n}{n}} dx
\]

\[
\leq c \sqrt{\frac{\log n}{n}} + c \sqrt{\frac{\log n}{n}}
\]

\[
\leq c \sqrt{\frac{\log n}{n}}
\]
Expected traffic through cell

- $E[\# \text{ of Lines passing through cell}] \leq c\sqrt{n \log n}$
- $E[\text{Traffic passing through cell}] \leq c\lambda(n)\sqrt{\log n}$

- **Question:** What is the actual traffic passing through cell?
- **Question:** What is the actual number of lines through each cell?
Lemma

\( \mathcal{D} = \) set of all disks of radius \( \zeta \)

\( C(D) = \) All great circles which intersect \( D \in \mathcal{D} \)

\[ \text{VC dim}(C(D): D \in \mathcal{D}) = \text{VC dim}(F(D): D \in \mathcal{D}) \]

where \( F(D) = \) Intersection of two large disks each larger than a hemisphere
The traffic through each cell

- VC dim (Set of sets) = VC dim (Set of complements)
- VC dim($G \cap G'$: $G \in G, G' \in G$) $\leq 10$ VC dim($G$)
- So VC-dim($C(D): D \in D$) $\leq 30$

- Hence

$$\Pr(\text{Every cell has less than } c\sqrt{n \log n} \text{ lines passing through it})$$

$$\geq 1 - \delta(n) \text{ where } \delta(n) \to 0$$

$$\Pr(\text{Every cell has traffic less than } c\lambda(n)\sqrt{n \log n} \text{ through it})$$

$$\geq 1 - \delta(n) \text{ where } \delta(n) \to 0$$
Feasible level of traffic

- Every cell has to handle \( c\lambda(n)\sqrt{n\log n} \) bits/sec
- Every cell can handle \( \frac{W}{1 + c_1} \) bits/sec
- So \( \lambda(n) \) can be handled if \( c\lambda(n)\sqrt{n\log n} \leq \frac{W}{1 + c_1} \)
- Theorem

\[
\Pr \left( \lambda(n) = \frac{c}{\sqrt{n\log n}} \text{ is feasible} \right) \rightarrow 1
\]
Upper bound on capacity: What is possible?

- Recall space constraint

- Space used per transmission
  \[ \leq \pi \Delta^2 r^2(n)/16 \]

- Total area available = 1

- Number of simultaneous transmissions
  \[ \leq \frac{16}{\pi \Delta^2 r^2(n)} \]

- Total transmitted by all nodes
  \[ \leq \frac{16W}{\pi \Delta^2 r^2(n)} \text{ bits/sec} \]
How much do we need?

- $E[\text{Length of line } L_i] = L$
- Mean number of hops per OD pair $\geq L/r(n)$
- There are $n$ OD pairs
- Total number of hops $\geq Ln/r(n)$
- Each OD pair requires $\lambda(n)$ bits/sec
- Total transmission required over all nodes $\geq Ln\lambda(n)/r(n)$

Feasibility condition

$$\frac{Ln\lambda(n)}{r(n)} \leq \frac{16W}{\pi \Delta^2 r^2(n)}$$

Upper bound

$$\lambda(n) \leq \frac{16W}{\pi \Delta^2 Ln r(n)} \text{ bits/sec} \quad \text{Make } r(n) \text{ small}$$
Connections based on distance
[Penrose ’97, Gupta & K ’98]

- **Model**: $n$ i.i.d. points in an unit square or disk of unit area

- **Connection rule**: Connect each node to every node that is within distance $r(n)$

- **Theorem**

\[
\lim_{n \to \infty} \text{Prob}(\text{Network is connected}) = 1 \text{ if and only if } \\
\pi r^2(n) = \frac{\log(n) + f(n)}{n} \text{ with } f(n) \to \infty
\]

- On average, each node should connect to more than $1 \cdot \log(n)$ neighbors
Network disconnectivity at small $r(n)$

- $n$ iid, uniformly located nodes
- Join two nodes by an edge if distance less than $r(n)$
- $G(n, r(n)) =$ resulting random graph
- Let $P(n, r(n)) =$ Probability of an isolated node

Theorem

Consider $\pi r^2(n) = \frac{\log n + k(n)}{n}$. Then $\lim P(n, r(n)) > 0$ if and only if $\limsup k(n) < +\infty$.

Also condition for connectedness of random geometric graph.
Asymptotic probability of isolated node

**Idea of proof**

\[ P(n, r(n)) \geq \sum_{i=1}^{n} \Pr(i \text{ is the only isolated node}) \]

\[ \geq \sum_{i=1}^{n} \Pr(i \text{ is isolated}) - \sum_{i=1}^{n} \sum_{j \neq i} \Pr(i \text{ and } j \text{ are isolated}) \]

\[ \geq n(1-A(r))^{n-1} - n(n-1)((A(2r) - A(r))(1 - \frac{3}{2}A(r)))^{n-2} \]

\[ + (1 - A(2r))(1 - 2A(r))^{n-2} \]
Upper bound on capacity

- We thus need \( r(n) \geq \sqrt{\frac{\log n}{\pi n}} \) for feasibility

- Thus

\[
\lambda(n) \leq \frac{16W}{\Delta^2 L \sqrt{\pi n \log n}} \text{ bits/sec}
\]
Capacity of randomly formed wireless networks

- **Theorem**

  \[
  \text{Capacity } \lambda(n) = \Theta\left(\frac{1}{\sqrt{n \log n}}\right) \text{ bits/sec}
  \]

- **Note**: Worse by factor \( \frac{1}{\sqrt{\log n}} \) than optimally designed networks

- Results also hold on plane: Use inversion map!
Physical Model: Random Network

- $n$ nodes randomly located
  - Each node chooses random destination
  - Equal throughput $\lambda$ bits/sec for all OD pairs
  - Each node chooses same power level $P$

- Theorem

$$\Theta \left( \frac{1}{\sqrt{n \log n}} \right) \leq \lambda(n) \leq \Theta \left( \frac{1}{\sqrt{n}} \right) \text{ bits/sec}$$

- With best choice of routes, hops, spatio-temporal scheduling

- Franceschetti, Dousse, Tse, Thiran '07:
  - In the Generalized Physical Model: $\Theta \left( \frac{1}{\sqrt{n}} \right)$

(Gupta and K '00)
Some Experimentation: A Scaling Law
Some experimentation: A scaling law

- Throughput = \(2.6/n^{1.68}\) Mbps per node
  - No mobility
  - No routing protocol overhead
    - Routing tables hardwired
  - No TCP overhead
    - UDP
  - IEEE 802.11

- Why \(1/n^{1.68}\)?
  - Much worse than optimal capacity = \(c/n^{1/2}\)
  - Worse even than \(1/n\) timesharing
  - Perhaps overhead of MAC layer?

\[ \log(\text{Thpt}) \]

\[ \log(\text{Number of Nodes}) \]
Summarizing ...
Why multi-hop?

- Multi-hop increases traffic carrying capacity
- It may also increase delay
Optimal operation under “collision” model

- Order-optimal spatial reuse architecture
  - Group nodes into cells of size $\log n$
  - Choose a common power level for all nodes
    » Nearly optimal
  - Power should be just enough to guarantee network connectivity
    » Sufficient to reach all points in neighboring cell
  - Route packets along nearly straight line path from cell to cell
Implications for designers

- Design networks with few nodes, or scaled down bandwidth, or support mainly nearest neighbor communications.

- Splitting into several sub-channels (TDMA, FDMA, CDMA) does not help in increasing capacity.

- Power consumption: Busy fraction of modems is $\Theta\left(\frac{1}{\log n}\right)$.

- Range of transmissions: Scaled length of hops is $\Theta\left(\sqrt{\frac{\log n}{n}}\right)$.

- Architecture for providing optimal capacity:
  Group nodes into cells of size $O(\log n)$ - one node in each cell serving as relay.

- $kn$ randomly placed relay nodes increase capacity by factor $\sqrt{k}$.

- Directed transmissions will help - space is a valuable resource - but only by a constant factor.
References - 1


References-2


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