Capacity Bounds for Ad hoc and Hybrid Wireless Networks*

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ABSTRACT

We study the capacity of static wireless networks, both ad hoc and hybrid, under the Protocol and Physical Models of communication, proposed in [1]. For ad hoc networks with n nodes, we show that under the Physical Model, where signal power is assumed to attenuate as $1/r^{\alpha}$, $\alpha > 2$, the transport capacity scales as $\Theta(\sqrt{n})$ bit-meters/sec. The same bound holds even when the nodes are allowed to approach arbitrarily close to each other and even under a more generalized notion of the Physical Model wherein the data rate is Shannon's logarithmic function of the SINR at the receiver. This result is sharp since it closes the gap that existed between the previous best known upper bound of $O(n^{\frac{\alpha-1}{\alpha}})$ and lower bound of $\Omega(\sqrt{n})$.

We also show that any spatio-temporal scheduling of transmissions and their ranges that is feasible under the Protocol Model can also be realized under the Physical Model by an appropriate choice of power levels for appropriate thresholds. This allows the generalization of various lower bound constructions from the Protocol Model to the Physical Model. In particular, this provides a better lower bound on the best case transport capacity than in [1].

For hybrid networks, we consider an overlay of μn randomly placed wired base stations. It has previously been shown in [6] that if all nodes adopt a common power level, then each node can be provided a throughput of at most $\Theta(\frac{1}{\log n})$ to randomly chosen destinations. Here we show that by allowing nodes to perform power control and properly choosing μ, it is further possible to provide a throughput of $\Theta(1)$ to any fraction f, $0 < f < 1$, of nodes. This result holds under both the Protocol and Physical models of communication. On the one hand, it shows that that the aggregate throughput capacity, measured as the sum of individual throughputs, can scale linearly in the number of nodes. On the other hand, the result underscores the importance of choosing minimum power levels for communication and suggests that simply communicating with the closest node or base station could yield good capacity even for multihop hybrid wireless networks.

Keywords

Wireless networks, ad hoc networks, hybrid networks, physical model, protocol model, throughput capacity, transport capacity.

1. INTRODUCTION

Asymptotic bounds on the capacity of ad hoc wireless networks have been established in [1]. Two models of communication were proposed. The *Protocol Model* requires that each receiver lie outside the interference region of every other transmitter. Here each transmitter transmitting to a receiver at distance r forms an interference region consisting of a disc of radius $(1+\Delta)r$ centered around itself. Under the alternate *Physical Model* of communication, the signal to interference plus noise ratio (SINR) is required to be above a pre-specified threshold β, where the signal power is assumed to decay as $r^{-\alpha}$ with distance r, for $\alpha > 2$. Further, two metrics for capacity are also defined in [1]. *Transport capacity* is defined as the total bit-distance product per second that can be transported by the network. *Throughput capacity*, on the other hand, is defined as the maximum common throughput that can be provided to each node with a randomly chosen destination.

Under the Protocol Model, upper and lower bounds in bit-meters/sec of $\sqrt{\frac{8}{\pi}}\frac{1}{\Delta}W\sqrt{n}$ and $\frac{W}{1+2\Delta}\frac{n}{\sqrt{n}+\sqrt{8\pi}}$, respectively, are obtained for the best case transport capacity in [1], where W is the capacity of the wireless channel in bits/sec. For random networks, i.e., networks where the node are dis-

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tributed uniformly and independently, and a destination is randomly chosen for each node, the upper bound on the per node throughput capacity is shown to be $\frac{c''W}{\Delta^2\sqrt{n\log n}}$ under the Protocol Model, for some constant c''. For the Physical Model, an upper bound of $O(n^{\frac{\alpha-1}{\alpha}})$ on the transport capacity has been established in [1]. The same bound is obtained in [7] for a Generalized Physical Model, where data rate is the Shannon logarithmic function of the SINR. However the best lower bound obtained has been $\Omega(\sqrt{n})$, leaving a gap between the upper and the lower bounds.

Information theoretic approaches have also been used to determine scaling laws on the total transport capacity of the network. In [3], assuming that the internode distance is lower bounded, and $\alpha > 6$, it is shown that the transport capacity is asymptotically bounded by the sum of the transmit power of the nodes in the network. Thus for domains of size $\Theta(n)$, transport capacity scales as $\Theta(n)$. This result applies to the Physical Model too. However, it is not known if the same bound holds in general for $\alpha > 2$, or when the nodes are allowed to approach arbitrarily close to one another.

For the case of the Protocol Model, it has been shown in [1] that discs of radius $\frac{\Delta r}{2}$ around the receivers are *exclusion regions*, i.e., are mutually disjoint from each other, where r is the distance from the transmitter to the particular receiver. In [2], larger exclusion regions have been obtained, thus improving the capacity bounds for the Protocol Model. An upper bound of $\sqrt{\frac{8}{\pi}}\frac{W\sqrt{n}}{\sqrt{(1+\Delta)\sqrt{\Delta}\sqrt{2+\Delta}}}$, and also an improved lower bound of $\sqrt{\frac{1}{\pi}}\frac{W\sqrt{n}}{\sqrt{(1+\Delta)\sqrt{\Delta}\sqrt{2+\Delta}}}$ bit-meters/sec are obtained. Thus the best achievable transport capacity has been bracketed to within a factor of $\sqrt{8} \approx 2.83$, irrespective of Δ, characterizing it fairly sharply.

In this paper, we first show that Protocol Model is a more restrictive model in comparison to the Physical Model in the following sense. Specifically, we prove that configurations satisfying the constraints of the protocol model for suitable choices of Δ also satisfy the constraints of an appropriate physical model. This immediately establishes that lower bound constructions under the protocol model continue to hold under the physical model. In particular, the grid of ellipses construction for the Protocol Model in [2] works for the Physical Model also. This provides a sharper lower bound on the best known transport capacity under Physical Model.

Further, we improve the upper bound for the Physical Model to $\Theta(\sqrt{n})$, thus establishing a sharp order of the transport capacity. The same bound holds even for the Generalized Physical Model mentioned before. Besides, our proof does not impose any lower bounds on the internode distance, thus allowing the result to hold for both dense and sparse networks. For a network with n nodes, if the domain size is $\Theta(n)$, the upper bound on transport capacity is $\Theta(n)$ for any $\alpha > 2$ and with the constraint that the maximum power level is bounded by $\Theta(n^{\alpha})$. Also the capacity is achievable by a simple grid arrangement of nodes. Thus for the special case of the Generalized Physical Model, we are able to characterize the scaling law for all $\alpha > 2$, unlike the information theoretic setting [3].

In [4], it has recently been shown that the maximum common throughput that can be furnished is $\Theta(\frac{1}{\sqrt{n}})$ when power control can be exercised with some nodes choosing lower power and some higher. Our best case upper bound of $\Theta(\frac{1}{\sqrt{n}})$ thus establishes that random networks are best case up to order.

Next we explore the throughput capacity of hybrid wireless networks. For the case of ad hoc networks, it is shown in [1] that throughput capacity scales as $\Theta(\frac{1}{\sqrt{n\log n}})$, which tends to zero as the number of nodes increases. In [5], it is shown that with n wireless nodes, the number of base stations must scale as at least $\Theta(\sqrt{n})$ to achieve better scaling of capacity than pure ad hoc networks. In [6], it is shown that for networks where the ratio of wireless nodes to base stations is bounded above by some constant, a throughput capacity of $\Theta(\frac{1}{\log n})$ is the optimal achievable. An important assumption here is that all nodes choose a common power level.

We show that by employing power control in the network, one can further improve the throughput capacity. For a random network with n wireless nodes and μn base stations, $\mu > 0$, we show that one can additionally provide a throughput of $\Theta(1)$ to some $\Theta(n)$ nodes with probability approaching 1 as $n \to \infty$. This result shows that the total capacity of random hybrid networks, measured as the sum of the throughputs furnished to the nodes, can scale linearly with the number of nodes. Further any fraction f, $0 < f < 1$, of nodes can be furnished this throughput provided that μ is sufficiently large. Also the scheme which achieves this requires nodes to communicate with their closest base stations thus saving battery power and underscoring the importance of using minimum power levels for communication.

Some of the implications for the design and operation of wireless networks are the following. Much current technology for decoding packets is essentially based on the Physical Model, where interference is treated as noise, rather than on using sophisticated multi-user detection where multiple interfering signals are simultaneously decoded at a node. For all wireless communication networks formed by using such technology, the present paper characterizes sharply the order of what is achievable. A class of application scenarios of much current interest is sensor networks. There, at present the emphasis is on using very low cost nodes where, currently at least, sophisticated multi-user detection is not envisaged. By allowing nodes to be arbitrarily closely placed, we attempt to capture deployments of high density, as for example in sensor networks as well as closely spaced personal area networks. At the other end, for hybrid networks, our results underline the importance of building power control into the protocol stack. By using power control, we show that the throughput furnished to any fraction of users can be improved in order. Moreover, since the strategy achieving this consists of communicating with nearby nodes and using them for relaying, it suggests that protocols should be designed to be efficient in the regime where the number of hops is large. Thus, attempts should be made to reduce all per hop costs. This will become increasingly important for large scale deployments of wireless networks as well as hybrid networks.

2. NETWORK MODEL

Consider a network of n nodes in a disc of unit area. Let X_i, $1 \leq i \leq n$, denote the location of node i. We will use X_k to denote a node as well as its location. Let $\{(X_k, X_{R(k)}) : k \in T\}$ be the set of all active transmitter-receiver pairs in some particular slot. As in [1], we assume a slotted model for convenience of exposition. Let the transmission radius, $|X_k - X_{R(k)}|$, be denoted as r_k. We first describe the models of communication.

2.1 The Protocol Model

The transmission from node X_i, $i \in T$, is successfully received by the receiver $X_{R(i)}$ only if

$$|X_k - X_{R(i)}| \geq (1+\Delta)|X_k - X_{R(k)}|, \quad (1)$$

for every $k \in T \backslash i$. Here $\Delta > 0$ models a guard zone around the transmission region. Equivalently, one can consider a guard zone around the receiver as in [1].

Generalized Protocol Model

As before, let $\{(X_k, X_{R(k)}) : k \in T\}$ denote the set of all active transmitter receiver pairs. Associated with each transmitter-receiver pair, $(X_k, X_{R(k)})$, $k \in T$, let there be an *interference region*, I_k, and assume that $X_{R(k)} \in I_k$. Suppose also that the necessary condition for a transmission from X_k to $X_{R(k)}$ to be successful is

$$X_{R(k)} \notin I_j, \forall j \in T, j \neq k. \quad (2)$$

Also no node can serve simultaneously as a receiver as well as a transmitter. Such a general interference footprint I_k can be used to model, for example, directional antennas.

2.2 The Physical Model

Suppose signal power suffers an attenuation by a factor of $\frac{1}{r^\alpha}$ as it traverses a distances r, where $\alpha > 2$ represents the path loss exponent. Then the received signal-to-interference-plus-noise ratio (SINR) for the transmission from X_k to $X_{R(k)}$ is

$$\frac{\frac{P_k}{|X_k - X_{R(k)}|^\alpha}}{N_0 + \sum_{i \in T, i \neq k} \frac{P_i}{|X_i - X_{R(k)}|^\alpha}}, \quad (3)$$

where N_0 represents the ambient noise. Under the physical model of communication, the transmission from X_k to $X_{R(k)}$ is successful only if its SINR is greater than β.

Generalized Physical Model

The Physical Model assumes a threshold based channel, where the signal can be decoded at a constant specified rate W bits/sec only if the SINR is greater than some threshold, failing which no throughput is received at all. Thus the corresponding throughput is either 0 or W bits/s. Here we generalize this notion as in [7] and assume that the throughput is a function of the SINR at the receiver. We use Shannon's capacity formula for the additive white Gaussian noise channel [8]. Thus the data rate for the transmitter receiver pair $(X_k, X_{R(k)})$ is given in bits/sec by

$$W_k = H_m \log_2\left(1 + \frac{\frac{P_k}{|X_k - X_{R(k)}|^\alpha}}{N_0 H_m + \sum_{i \in T, i \neq k} \frac{P_i}{|X_i - X_{R(k)}|^\alpha}}\right) \quad (4)$$

where H_m is the bandwidth of channel m in hertz, and $\frac{N_0}{2}$ is the noise spectral density in watts/hertz.

3. CAPACITY BOUNDS FOR AD HOC WIRELESS NETWORKS UNDER THE PROTOCOL MODEL

In [1], upper bounds on the best case transport capacity under the Protocol Model are derived. It uses the fact that the $\frac{\Delta r}{2}$ neighborhood around a receiver of a transmission of range r is an *exclusion region*. Formally, an exclusion region is an area associated with an active receiver that must remain disjoint from the exclusion region of every other active receiver in the network at that time, for any configuration of transmitters and receivers.

In [2], it is shown that these *exclusion regions* are much larger, and can be computed for arbitrary interference footprints including those arising from directional antennas. Specifically, for the Generalized Protocol Model, Theorem 2.2 in [2] states that the set $E_k = \{P : |P - X_{R(k)}| < |P - Q|, \forall Q \notin I_k\}$ is a valid exclusion region for each receiver $R(k)$. Further, Theorem 3.3 states this set is always convex. For the case of Protocol Model, i.e., where I_k is a disk of radius $(1+\Delta)|X_i - X_{R(i)}|$ centered at X_i, the set E_k corresponds to an ellipse with X_k and $X_{R(k)}$ as the foci, and eccentricity $\frac{1}{1+\Delta}$, i.e., $E_k = \{P : |P - X_{R(k)}| + |P - X_k| \leq (1+\Delta)|X_{R(k)} - X_k|\}$.

Using these exclusion regions, an upper bound in bit-meters/sec of $\sqrt{\frac{8}{\pi}} \frac{W\sqrt{n}}{\sqrt{(1+\Delta)\sqrt{\Delta}\sqrt{2+\Delta}}}$ is derived for the best case transport capacity. Further a simple grid of ellipses configuration allows a lower bound construction which achieves $\sqrt{\frac{1}{\pi}} \frac{W\sqrt{n}}{\sqrt{(1+\Delta)\sqrt{\Delta}\sqrt{2+\Delta}}}$ bit-meters/sec. Thus, carefully characterizing the exclusion regions allows bracketing the best case transport capacity for Protocol Model to within a factor of $\sqrt{8}$. Similarly, for random networks where node locations are chosen uniformly and independently, it is shown that the throughput capacity is bounded by $\frac{c''W}{(1+\Delta)\sqrt{\Delta}\sqrt{2+\Delta}\sqrt{n \log n}}$, where c'' is some constant.

4. CAPACITY BOUNDS FOR AD HOC WIRELESS NETWORKS UNDER THE PHYSICAL MODEL

In this section, we bound the best case transport capacity of wireless networks under the two Physical Models discussed in Section 2. We begin by showing how the Protocol Model compares with these two models.

4.1 Physical Model Compared with Protocol Model

The constraints defined by the Protocol Model are local. They only require certain localities of transmitters to be free of receivers. On the other hand, the Physical Model considers the cumulative interference due to all the nodes in the network. Thus, intuitively it appears that the Physical Model is a much more restrictive model, and would entail lower capacity. However, contrary to intuition, it turns out that it is in fact the Protocol Model that can be considered as the more restrictive model, as the next theorem shows.

THEOREM 4.1. *Given any $\beta > 0$, there is a $\Delta_\beta > 0$, such that for any $\Delta > \Delta_\beta$, if any configuration $\{(X_k, X_{R(k)}), k \in T\}$ satisfies the Protocol Model with guard parameter Δ, then there is an assignment of powers, $\{P_k, k \in T\}$, such that the configuration also satisfies the constraints of the Physical Model with SINR threshold β. In particular, $\Delta_\beta \le (2c_\alpha\beta)^{1/\alpha}$, where $c_\alpha := \frac{24}{\alpha-2}2^{\alpha-2}$.*

Proof: From [2], we know that the exclusion region for the pair $(X_k, X_{R(k)})$ is a region bounded by the following ellipse:

$$E_k = \{P : |P - X_{R(k)}| + |P - X_k| = (1+\Delta)|X_{R(k)} - X_k|\}.$$

In particular, let D_k and D'_k denote disks of radius $\frac{\Delta r_k}{2}$ around X_k and $X_{R(k)}$ respectively. Then both these disks are contained in the ellipse. Thus, for any distinct i and j, D_i is mutually disjoint with D_j and D'_j.

Let the ambient noise be N_0. We will show that a power assignment that works is $P_k = c\Delta^2 r_k^2$, where $c \ge \frac{N_0}{c_\alpha}(\frac{\Delta}{\sqrt{\pi}})^{\alpha-2}$. To show this, we first bound the interference at any receiver due to all the other transmitters. Let $D_{ij} := \{x : x \in D_i,\ |x - X_{R(j)}| \le |X_i - X_{R(j)}|\}$. Thus D_{ij} represents the part of D_i which is closer to the receiver $X_{R(j)}$ than the transmitter X_i. Consider a circle centered at $X_{R(j)}$ and passing through X_i, and let it cut the boundary of D_i at A and B. Then, since $|X_{R(j)} - X_i| = |X_{R(j)} - A| \ge |A - X_i|$, $\angle A X_i X_{R(j)} \ge \pi/3$. Similarly, $\angle B X_i X_{R(j)} \ge \pi/3$, Thus the area of D_{ij} is at least one third of the area of D_i. Let I_k denote the interference at receiver $X_{R(k)}$. We now have,

$$\begin{aligned}
I_k &:= \sum_{i\in T, i\ne k} \frac{P_i}{|X_i - X_{R(k)}|^\alpha} \\
&= \sum_{i\in T, i\ne k} \frac{4c}{\pi}\int_{D_i} \frac{dA}{|X_i - X_{R(k)}|^\alpha} \\
&\le \sum_{i\in T, i\ne k} \frac{12c}{\pi}\int_{D_{ik}} \frac{dA}{|X_i - X_{R(k)}|^\alpha} \\
&\le \sum_{i\in T, i\ne k} \frac{12c}{\pi}\int_{D_{ik}} \frac{dA}{|x - X_{R(k)}|^\alpha} \\
&\quad \text{(where } x \text{ is the position vector of element } dA) \\
&= \frac{12c}{\pi}\int_{S_k} \frac{dA}{|x - X_{R(k)}|^\alpha} \\
&\quad \text{(where } S_k := \cup_{i\in T, i\ne k} D_{ik}) \\
&\le \frac{12c}{\pi}\int_{|x - X_{R(k)}| \ge \Delta r_k/2} \frac{dA}{|x - X_{R(k)}|^\alpha} \\
&\quad \text{(since } S_k \cap D_k = \phi) \\
&= \frac{12c}{\pi}\int_{\Delta r_k/2}^{\infty} \frac{2\pi r dr}{r^\alpha} \\
&= \frac{24c}{\alpha-2}\left(\frac{2}{\Delta r_k}\right)^{\alpha-2} \\
&= c c_\alpha (\Delta r_k)^{2-\alpha}.
\end{aligned}$$

Now the SINR for the pair $(X_k, X_{R(k)})$ is

$$\begin{aligned}
& \frac{\frac{P_k}{r_k^\alpha}}{N_0 + I_k} \\
\ge\ & \frac{c\Delta^2 r_k^{2-\alpha}}{N_0 + c c_\alpha(\Delta r_k)^{2-\alpha}} \\
=\ & \frac{\Delta^\alpha}{\frac{N_0}{c}\Delta^{\alpha-2} r_k^{\alpha-2} + c_\alpha} \\
\ge\ & \frac{\Delta^\alpha}{2c_\alpha} \ \text{(since } r_k \le \tfrac{1}{\sqrt{\pi}}) \\
\ge\ & \beta.
\end{aligned}$$

□

Thus we have shown that any spatio-temporal scheduling, along with choices of ranges satisfying the constraints of the Protocol Model, also admits a choice of power levels which together with appropriate thresholds also satisfy the constraints of the Physical Model. This allows direct generalization of various lower bound constructions for the Protocol Model to the Physical Model.

In particular, by exploiting the *grid of ellipses* configuration in [2], which yields the largest lower bound on the best case transport capacity known to date (as far as the authors are aware) for the Protocol Model, we get the following lower bound result for the Physical Model. Note that it sharpens the lower bound for transport capacity under the Physical Model obtained in [1].

COROLLARY 4.1. *There is a placement of nodes inside a unit disk and an assignment of traffic patterns, such that, under the Physical Model with SINR threshold β, the network can achieve*

$$\sqrt{\frac{1}{\pi}}\frac{W\sqrt{n}}{\sqrt{(1+\Delta)\sqrt{\Delta}\sqrt{2+\Delta}}} \text{ bit-meters/sec,} \qquad (5)$$

where $\Delta := (2c_\alpha\beta)^{1/\alpha}$.

4.2 A Sharp Upper Bound on the Transport Capacity under the Generalized Physical Model

In this section, we will establish upper bounds on the best case transport capacity of ad hoc wireless networks under the Generalized Physical Model described in Section 2. Note that, asymptotically, the same bound holds for the Physical Model as well. To see this, note that for any given configuration of transmitter-receiver pairs, together with the choice of transmission powers, a particular receiver would receive W bits/sec of data under the Physical Model only when the SINR is greater than some threshold β. In that case, the Generalized Physical Model would also allow some constant throughput depending upon β. Thus, within a constant, the total bit-distance/sec obtained under the Generalized Physical Model would be at least as much as that obtained under the Physical Model. Taking maximum over all possible configurations, the transport capacity under the Generalized Physical Model is asymptotically lower bounded by that under the Physical Model.

We next show the sharp result that the bound is $\Theta(\sqrt{n})$, which is asymptotically the same as the lower bound in Corollary 4.1. This directly improves the previous bound of $n^{\frac{\alpha-1}{\alpha}}$ in [1], and provides an upper bound which coincides in order with the lower bound, thus sharply characterizing the scaling law.

We first establish some results that are used to prove the main result.

LEMMA 4.1. *For $x \in R^+$ and $\alpha \geq 1$, $\ln(1+x^\alpha) \leq \alpha x$.*

Proof:

$$\begin{aligned} & \ln(1+x^\alpha) \\ \leq \; & \ln((1+x)^\alpha) \\ = \; & \alpha \ln(1+x) \\ \leq \; & \alpha x. \end{aligned}$$

□

LEMMA 4.2. *Consider an $m \times m$ square grid with each grid box being a unit square. Let m be a power of 2. Let $S = \{Q_i, 1 \leq i \leq k\}$ be a set of k points with $k \leq m^2$, such that each square in the grid contains at most one point from S. Let there be a total ordering "$\prec$" imposed on the k points in S. For $1 \leq i \leq k$, define $d_i = min(\{m\sqrt{2}\} \cup \{|Q_j - Q_i| : 1 \leq j \leq k, j \neq i, Q_i \prec Q_j\})$, i.e., the distance to the nearest higher ordered point. Then $\sum_{i=1}^k d_i \leq 3\sqrt{2}m^2 - 2\sqrt{2}m$.*

Proof: Let $f(m)$ denote the upper bound on $\sum_{i=1}^k d_i$ over any choice of the points and their ordering. We will obtain a recurrence for $f(m)$. Note first that $f(1) = \sqrt{2}$. For $m \geq 4$, divide the grid into four equal quadrants of size $\frac{m}{2} \times \frac{m}{2}$. For any point Q_i, let $d'_i = min(\{\frac{m}{2}\sqrt{2}\} \cup \{|Q_j - Q_i| : 1 \leq j \leq k, j \neq i, Q_i$ and Q_j are in the same quadrant, $Q_i \prec Q_j\})$ be similarly defined with respect to the quadrant in which the point lies.

Next note that for any two points Q_i and Q_j in the same quadrant, if $Q_i \prec Q_j$, then $d_i \leq |Q_i - Q_j| \leq \frac{m}{2}\sqrt{2}$, i.e., at least one of them has $d_i \leq \frac{m}{2}\sqrt{2}$. Thus, there can be at most one point in each quadrant with $d_i = m\sqrt{2}$. Such a point can only be the *maximal* element in the quadrant under the ordering "$\prec$". For such a maximal element, $d'_i = \frac{m}{2}\sqrt{2}$. Also, if Q_j is not the maximal element in the quadrant, then $d_j \leq d'_j$ since d_j involves taking minimum over a larger set.

We now have $\sum_{i=1}^k d_i \leq \sum_{i=1}^k d'_i + 4(m\sqrt{2} - \frac{m}{2}\sqrt{2})$. Taking maximum over all configurations, we get

$$f(m) \leq 4f(m/2) + 2m\sqrt{2}.$$

The base condition is $f(1) = \sqrt{2}$. We can show by induction that the solution of this recurrence is $f(m) \leq (3\sqrt{2}m^2 - 2\sqrt{2}m)$. For $m = 1$, the base condition is satisfied. For $m \geq 2$, we get

$$\begin{aligned} f(m) & \leq 4f(m/2) + 2m\sqrt{2} \\ & \leq 4(3\sqrt{2}\frac{m^2}{4} - 2\sqrt{2}\frac{m}{2}) + 2m\sqrt{2} \\ & = 3\sqrt{2}m^2 - 2\sqrt{2}m. \end{aligned}$$

□

We are now ready to present our main result.

THEOREM 4.2. *Consider the Generalized Physical Model with $\alpha > 2$, noise spectral density $N_0/2$, and available bandwidth H hertz, that can be shared by M sub-channels, $m = 1, \ldots, M$ with bandwidths H_m satisfying $\sum_{m=1}^M H_m = H$. Suppose that the maximum power level P_{max} used over a sub-channel m with bandwidth H_m hertz is bounded by $P_{max} \leq H_m N_0 n^{\frac{\alpha}{2}}$, where n is the number of nodes in a square of unit area. Then the transport capacity, for any arbitrary placement of nodes and choice of transmission powers, is upper bounded by*

$$H k_\alpha \sqrt{n} \text{ bit-meters/sec,} \tag{6}$$

where $k_\alpha = \log_2(e)(\alpha(13\sqrt{2}+6) + 4((\sqrt{2}+1)2)^\alpha)$.

Proof: We will assume a slotted operation with slot length τ secs. As earlier, let $\{(X_k, X_{R(k)}) : k \in T\}$ be the set of all active transmitter-receiver pairs. Let $r_k := |X_k - X_{R(k)}|$ and $r_{ki} := |X_k - X_{R(i)}|$. Let P_i be the power level used by node i, and let w be a node transmitting at power level $\max_{i \in T}\{P_i\}$. Let $T' = T \backslash \{w\}$. Then the total traffic in bit-meters for slot s and channel m, with W_i denoting the data rate obtained by the i^{th} transmitter, is given by:

$$\begin{aligned} & \sum_{i \in T} r_i W_i \tau \\ = \; & \sum_{i \in T} r_i \tau H_m \log_2 (1 + \frac{\frac{P_i}{r_i^\alpha}}{N_0 H_m + \sum_{k \in T, k \neq i} \frac{P_k}{r_k i^\alpha}}) \\ \leq \; & r_w \tau H_m \log_2 (1 + \frac{\frac{P_w}{r_w^\alpha}}{N_0 H_m}) + \\ & \sum_{i \in T'} r_i \tau H_m \log_2 (1 + \frac{\frac{P_i}{r_i^\alpha}}{\sum_{k \in T, k \neq i} \frac{P_k}{r_{ki}^\alpha}}) \\ \leq \; & \tau H_m \log_2(e)(\alpha(\frac{P_w}{N_0 H_m})^{1/\alpha} + \\ & \sum_{i \in T'} r_i \ln (1 + \frac{\frac{P_i}{r_i^\alpha}}{\sum_{k \in T, k \neq i} \frac{P_k}{r_{ki}^\alpha}})) \text{ (using Lemma 4.1)} \\ \leq \; & \tau H_m \log_2(e)(\alpha\sqrt{n} + \sum_{i \in T'} r_i \ln (1 + \frac{\frac{P_i}{r_i^\alpha}}{\sum_{k \in T, k \neq i} \frac{P_k}{r_{ki}^\alpha}})) \\ & \text{(using } P_w \leq P_{max} \leq N_0 H_m n^{\frac{\alpha}{2}}). \end{aligned} \tag{7}$$

We will now compute bounds on the sum in (7). For this, we first divide T' into disjoint classes $C_0, C_1, \ldots$ as follows. For $j \geq 0$, define $R_j := 2^{j - \lceil \log_2(\sqrt{n}) \rceil}$. Note that $\frac{2^{j-1}}{\sqrt{n}} < R_j \leq \frac{2^j}{\sqrt{n}}$. Let $C_0 := \{i : i \in T', |X_i - X_{R(i)}| < R_0\}$. For $j > 0$, define $C_j := \{i : i \in T', R_{j-1} \leq |X_i - X_{R(i)}| < R_j\}$.

Note that we can write the sum term on the RHS of (7) as

$$\sum_{i \in T'} r_i \ln\left(1 + \frac{\frac{P_i}{r_i^\alpha}}{\sum_{k \in T, k \neq i} \frac{P_k}{r_{ki}^\alpha}}\right)$$
$$= \sum_{j \geq 0} \sum_{i \in C_j} r_i \ln\left(1 + \frac{\frac{P_i}{r_i^\alpha}}{\sum_{k \in T, k \neq i} \frac{P_k}{r_{ki}^\alpha}}\right)$$

We will now determine $\sum_{i \in C_j} r_i \ln(1 + \frac{\frac{P_i}{r_i^\alpha}}{\sum_{k \in T, k \neq i} \frac{P_k}{r_{ki}^\alpha}})$ for each $j \geq 0$. To do this, we divide our square domain into a square grid with grid length R_j. Let us suppose that there are G_j grid squares each with at least one receiver, and let us denote them by Γ_g, $1 \leq g \leq G_j$, where $G_j \leq \frac{1}{R_j^2}$. Denote the number of transmitter-receiver pairs whose receiver lies in cell Γ_g by n_g. Let the transmitter-receiver distances and transmitted power levels for these pairs be denoted respectively by r_{gh} and P_{gh}, where $0 \leq h \leq n_g - 1$. Let the total interference at the receiver be denoted by I_{gh}. For each Γ_g, let Q_g denote the receiver corresponding to the pair with the highest transmitted power level among all pairs with receivers in Γ_g. Assume that r_{g0}, P_{g0} and I_{g0} correspond to this receiver Q_g . Then,

$$\sum_{i \in C_j} r_i \ln\left(1 + \frac{\frac{P_i}{r_i^\alpha}}{\sum_{k \in T, k \neq i} \frac{P_k}{{r_{ki}}^\alpha}}\right)$$
$$= \sum_{g=1}^{G_j} r_{g0} \ln\left(1 + \frac{\frac{P_{g0}}{{r_{g0}}^\alpha}}{I_{g0}}\right) + \sum_{g=1}^{G_j} \sum_{h=1}^{n_g - 1} r_{gh} \ln\left(1 + \frac{\frac{P_{gh}}{{r_{gh}}^\alpha}}{I_{gh}}\right). \quad (8)$$

Next consider all the maximum power receivers Q_g, $1 \leq g \leq G_j$. Define a total ordering " $\prec$ " by ordering them according to their transmitted power levels and breaking ties arbitrarily (so $P_{g0} < P_{h0} \Rightarrow Q_g \prec Q_h$). Define $d_g = \min(\{\sqrt{2}\} \cup \{|Q_h - Q_g| : 1 \leq h \leq G_j, h \neq g, Q_g \prec Q_h\})$.

First note that the node w, which is the transmitter using the globally maximum power level, is within a distance $\sqrt{2}$ of every Q_g, $1 \leq g \leq G_j$. Thus, from the definition of d_g, for any receiver Q_g, there must be a transmitter within a distance d_g of Q_g that is transmitting with power at least P_{g0}. Thus, the interference I_{g0} is at least

$$I_{g0} \geq \frac{P_{g0}}{(d_g + R_j)^\alpha}. \quad (9)$$

Similarly, we obtain a bound on I_{gh}, $1 \leq h \leq n_g - 1$. Corresponding to the grid box Γ_g, define

$$P_g := \sum_{h=1}^{n_g - 1} P_{gh}. \quad (10)$$

Then $I_{gh} \geq \frac{P_g - P_{gh}}{(\sqrt{2}R_j + R_j)^\alpha}$, by considering only those transmitters as interferers whose receivers are within the grid square Γ_g, and upper bounding the distance of such transmitters. Since for $1 \leq h \leq n_g$, $P_{g0} \geq P_{gh}$, we have, $P_g \geq 2P_{gh}$, or equivalently, $P_g - P_{gh} \geq \frac{P_g}{2}$. Thus, the interference I_{gh} is bounded as

$$I_{gh} \geq \frac{P_g/2}{(\sqrt{2}R_j + R_j)^\alpha}. \quad (11)$$

We will now use (9) and (11) to simplify (8). First, we have

$$\sum_{g=1}^{G_j} r_{g0} \ln\left(1 + \frac{\frac{P_{g0}}{{r_{g0}}^\alpha}}{I_{g0}}\right)$$
$$\leq \sum_{g=1}^{G_j} r_{g0} \ln\left(1 + \frac{\frac{P_{g0}}{{r_{g0}}^\alpha}}{\frac{P_{g0}}{(d_g + R_j)^\alpha}}\right) \text{ (using (9))}$$
$$\leq \sum_{g=1}^{G_j} \alpha(d_g + R_j) \text{ (using Lemma 4.1)}$$
$$\leq \alpha(R_j(3\sqrt{2}\frac{1}{{R_j}^2}) + R_j(G_j)) \text{ (using Lemma 4.2)}$$
$$\leq \frac{\alpha}{R_j}(3\sqrt{2} + 1). \quad (12)$$

Next note that we have

$$\sum_{g=1}^{G_j} \sum_{h=1}^{n_g - 1} r_{gh} \ln\left(1 + \frac{\frac{P_{gh}}{{r_{gh}}^\alpha}}{I_{gh}}\right)$$
$$\leq \sum_{g=1}^{G_j} \sum_{h=1}^{n_g - 1} r_{gh} \ln\left(1 + \frac{2P_{gh}}{P_g} \frac{((\sqrt{2}+1)R_j)^\alpha}{{r_{gh}}^\alpha}\right). \quad (13)$$

For $j = 0$, (13) simplifies as,

$$\sum_{g=1}^{G_0} \sum_{h=1}^{n_g - 1} r_{gh} \ln\left(1 + \frac{2P_{gh}}{P_g} \frac{((\sqrt{2}+1)R_0)^\alpha}{{r_{gh}}^\alpha}\right)$$
$$\leq \sum_{g=1}^{G_0} \sum_{h=1}^{n_g - 1} r_{gh} \ln\left(1 + \frac{((\sqrt{2}+1)R_0)^\alpha}{{r_{gh}}^\alpha}\right)$$
$$\text{(since } P_g \geq 2P_{gh}\text{)}$$
$$\leq \sum_{g=1}^{G_0} \sum_{h=1}^{n_g - 1} \alpha(\sqrt{2}+1)R_0 \text{ (using Lemma 4.1)}$$
$$= \alpha(\sqrt{2}+1)R_0 \sum_{g=1}^{G_0} \sum_{h=1}^{n_0 - 1} 1$$
$$\leq \alpha(\sqrt{2}+1)\frac{1}{\sqrt{n}}(n)$$
$$= \alpha(\sqrt{2}+1)\sqrt{n}. \quad (14)$$

For $j > 0$, (13) simplifies as,

$$\begin{aligned}
&\sum_{g=1}^{G_j}\sum_{h=1}^{n_g-1} r_{gh}\ln\left(1+\frac{2P_{gh}}{P_g}\frac{((\sqrt{2}+1)R_j)^\alpha}{{r_{gh}}^\alpha}\right)\\
\le\ &\sum_{g=1}^{G_j}\sum_{h=1}^{n_g-1} R_j\ln\left(1+\frac{2P_{gh}}{P_g}\frac{((\sqrt{2}+1)R_j)^\alpha}{{R_{j-1}}^\alpha}\right)\\
&\text{(since } R_{j-1}\le r_{gh}<R_j)\\
=\ &\sum_{g=1}^{G_j}\sum_{h=1}^{n_g-1} R_j\ln\left(1+\frac{2P_{gh}}{P_g}((\sqrt{2}+1)2)^\alpha\right)\\
&\text{(since } R_j=2R_{j-1})\\
\le\ &((\sqrt{2}+1)2)^\alpha R_j\sum_{g=1}^{G_j}\sum_{h=1}^{n_g-1}\frac{2P_{gh}}{P_g}\\
&\text{(since } \ln(1+x)\le x)\\
\le\ &((\sqrt{2}+1)2)^\alpha R_j\sum_{g=1}^{G_j}2 \text{ (using (10))}\\
\le\ &((\sqrt{2}+1)2)^\alpha R_j 2\left(\frac{1}{{R_j}^2}\right) \text{ (since } G_j\le\tfrac{1}{R_j^2})\\
=\ &((\sqrt{2}+1)2)^\alpha\frac{2}{R_j}. \qquad (15)
\end{aligned}$$

Finally, we can simplify the summation in (7) as follows:

$$\begin{aligned}
&\sum_{i\in T'} r_i\ln\left(1+\frac{\frac{P_i}{r_i^\alpha}}{\sum_{k\in T,k\ne i}\frac{P_k}{r_{ki}^\alpha}}\right)\\
=\ &\sum_{j\ge 0}\sum_{i\in C_j} r_i\ln\left(1+\frac{\frac{P_i}{r_i^\alpha}}{\sum_{k\in T,k\ne i}\frac{P_k}{{r_{ki}}^\alpha}}\right)\\
=\ &\sum_{j\ge 0}\sum_{g=1}^{G_j} r_{g0}\ln\left(1+\frac{\frac{P_{g0}}{{r_{g0}}^\alpha}}{I_{g0}}\right)+\\
&\sum_{j\ge 0}\sum_{g=1}^{G_j}\sum_{h=1}^{n_g-1} r_{gh}\ln\left(1+\frac{\frac{P_{gh}}{r_{gh}^\alpha}}{I_{gh}}\right)\\
\le\ &\sum_{j\ge 0}\frac{\alpha}{R_j}(3\sqrt{2}+1)+(\alpha(\sqrt{2}+1)\sqrt{n})+\\
&\sum_{j>0}\left(((\sqrt{2}+1)2)^\alpha\frac{2}{R_j}\right) \text{ (using (12), (14) and (15))}\\
=\ &\frac{\alpha}{R_0}(3\sqrt{2}+1)\sum_{j\ge 0}2^{-j}+(\alpha(\sqrt{2}+1)\sqrt{n})+\\
&\left(((\sqrt{2}+1)2)^\alpha\frac{2}{R_0}\right)\sum_{j>0}2^{-j} \text{ (since } R_j=2^jR_0)\\
<\ &\alpha 2\sqrt{n}(3\sqrt{2}+1)(2)+(\alpha(\sqrt{2}+1)\sqrt{n})+\\
&(((\sqrt{2}+1)2)^\alpha 4\sqrt{n})(1) \text{ (since } R_j>\tfrac{2^{j-1}}{\sqrt{n}})\\
=\ &\sqrt{n}(\alpha(13\sqrt{2}+5)+4((\sqrt{2}+1)2)^\alpha). \qquad (16)
\end{aligned}$$

Using (16), we now simplify (7). Thus,

$$\begin{aligned}
&\sum_{i\in T} r_iW_i\tau\\
\le\ &\tau H_m\log_2(e)(\alpha\sqrt{n}+\sqrt{n}(\alpha(13\sqrt{2}+5)+\\
&4((\sqrt{2}+1)2)^\alpha))\\
=\ &\tau H_m\log_2(e)\sqrt{n}(\alpha(13\sqrt{2}+6)+4((\sqrt{2}+1)2)^\alpha)\\
=\ &\tau H_m k_\alpha\sqrt{n}.
\end{aligned}$$

Now assume that the transmission lasts for a total time of S sec. The total capacity in bit-meters/sec is then,

$$\begin{aligned}
&\frac{1}{S}\sum_{s=1}^{S/\tau}\sum_{m=1}^{M}\sum_{i\in\tau} r_iW_i\tau\\
\le\ &\frac{1}{S}\sum_{s=1}^{S/\tau}\sum_{m=1}^{M}(\tau H_m k_\alpha\sqrt{n})\\
=\ &\tau k_\alpha\sqrt{n}\frac{1}{S}\sum_{s=1}^{S/\tau}\sum_{m=1}^{M}H_m\\
=\ &\tau k_\alpha\sqrt{n}\frac{1}{S}\sum_{s=1}^{S/\tau}H\\
=\ &Hk_\alpha\sqrt{n}.
\end{aligned}$$

□

Note that our upper bound holds even when the nodes are allowed to approach arbitrarily close to each other, without requiring a lower bound on internode distance. This is generally done to avoid the divergence of the signal power attenuation function at $r = 0$. However, the lower bound construction does not depend on this divergence. Thus our bounds hold for dense as well as expanding networks. By considering a domain of size A and shrinking it to domain of unit size using the mapping $P'_i = P_iA^{-\frac{\alpha}{2}}$ for the power levels, we get the following corollary:

COROLLARY 4.2. *For an ad hoc network with n nodes in a domain with area A, under the Physical Model with path loss exponent $\alpha > 2$, maximum power level bounded by $P_{max} \le H_mN_0(nA)^{\frac{\alpha}{2}}$, for any channel with bandwidth H_m and Noise spectral density $\frac{N_0}{2}$, the transport capacity scales as $\Theta(\sqrt{An})$ when each node has a power constraint. In particular, for networks with domain area $\Theta(n)$, the transport capacity grows as $\Theta(n)$.*

5. CAPACITY BOUNDS FOR HYBRID NETWORKS WITH POWER CONTROL

In this section, we consider the throughput capacity of static random wireless networks with an overlay of randomly placed base stations connected by high throughput connections. Our goal is to show that in addition to providing a throughput of $\Theta(\frac{1}{\log n})$ to each node, one can provide a throughput of $\Theta(1)$ to a guaranteed fraction of the nodes. This result shows that the transport capacity of random hybrid networks scales linearly in the number of nodes.

We consider a network formed by n wireless nodes and μn base stations, $\mu > 0$, in a disk D of unit area. Both the base stations and the wireless nodes are assumed to be randomly distributed uniformly in the disk. The base stations are assumed to be connected to each other with a very high bandwidth network so that there are no bottlenecks associated with the base stations. Further, the wireless nodes can communicate with the base stations and with each other using omni-directional antennas. Let X_k, $1 \leq k \leq n(1+\mu)$, denote the location of node k, where the node may either be a base station or a wireless node. We consider both the Protocol Model and the Physical Model for transmission as defined in Section 2. We will assume that nodes can perform power control in contrast to [6]. That is, nodes can arbitrarily choose their transmission radii in the Protocol Model, or their power levels in the Physical Model.

5.1 A Spatial Tessellation

We now proceed to constructively show how to provide $\Theta(1)$ throughput to $\Theta(n)$ nodes. We begin by considering a spatial tessellation of the disk. We require each cell resulting from the tessellation to contain exactly one wireless node. Nodes in each cell will be allowed to communicate only with a base station in their respective cells. A natural tessellation scheme considered here is the Voronoi tessellation of the disk [9] using the base stations as generators. Briefly, a Voronoi tessellation of a set S with generator set $\{a_1, a_2, \ldots, a_n\}$ assigns cell $V(a_i)$ to point a_i, $1 \leq i \leq n$, where

$$V(a_i) = \{x \in S : |x - a_i| = min_{1 \leq j \leq n}|x - a_j|\}. \quad (17)$$

We first develop some useful properties of this tessellation.

DEFINITION 5.1. *Call a Voronoi cell $V(a_i)$, η-fat, if an open disk of area η centered at a_i does not intersect with the Voronoi cell of any other generator. Call the cell η-thin if it is not η-fat. Let $t_i := \sup\{\eta : \eta > 0,\ V(a_i) \text{ is } \eta\text{-fat}\}$, denote the* fatness *of cell $V(a_i)$.*

Next, we establish a result about the "total fatness" of any set of cells, i.e., the sum of the fatness of each cell in a set of cells.

LEMMA 5.1. *Consider the Voronoi tessellation of a unit disk D using n uniform iid points as generators. Then, for $0 < f < 1$, $c < \frac{f^2}{c_1}$ and $c_1 = 13^2 e(\pi + 4\sqrt{\pi})(2^{1+\frac{4}{\pi}})$,*

$$\lim_{n \to \infty} Prob(\exists\ fn \text{ cells with total fatness} \leq c) = 0. \quad (18)$$

Proof: Let the set of n generators be $G := \{a_1, a_2, \ldots, a_n\}$, with a_i chosen randomly in a uniform iid fashion. Let $M(a_i)$ denote the point in $G \backslash \{a_i\}$ closest to a_i. Note that the function M maps at most 6 points to the same value. To see why, note that if $M(a_i) = M(a_j) = a_k$, then we have $|a_i - a_k| \leq |a_i - a_j|$ and $|a_j - a_k| \leq |a_i - a_j|$. Thus $\angle a_i a_k a_j \geq \frac{\pi}{3}$, and consequently a_k can serve as the closest point to at most six other points.

Next we construct a lower bound for the fatness of a cell in terms of the function M. Note first that if $M(a_i) = a_j$ and C is a disk centered at a_i with radius $\frac{|a_j - a_i|}{2}$, then $C \cap D \subseteq V(a_i)$. Hence, C does not intersect with any other cell. Thus the fatness of $V(a_i)$ is at least the area of C. In general, for any set $G' \subseteq G$, the total fatness of all cells corresponding to generators in G' is at least

$$\sum_{a_k \in G'} \frac{\pi |a_k - M(a_k)|^2}{4}. \quad (19)$$

Suppose now that there exist fn cells whose total fatness is at most c, and let the set of corresponding generators be $B := \{b_1, b_2, \ldots, b_{fn}\}$. Consider the largest set S of pairs $(c_k, M(c_k))$, such that $c_k \in B$ and no two pairs in S share the same points, i.e., the images under M are distinct and the image set is disjoint from the domain set. Let $r := |S|$. Then

$$r \geq \frac{fn}{13}. \quad (20)$$

This is because, given that $(c_k, M(c_k))$ is in S, a pair $(c_j, M(c_j))$ will be eliminated only if $M(c_k) = c_j$ (at most one such pair possible) or $M(c_k) = M(c_j)$ (at most five such pairs possible) or $c_k = M(c_j)$ (at most six such pairs possible). Thus at most twelve other pairs will be eliminated.

In the following, let $r := \frac{fn}{13}$. We can now bound the probability in (18) as follows:

$$\begin{aligned}
& Prob(\exists\ fn \text{ cells with total fatness} \leq c) \\
\leq\ & Prob(\exists \text{ distinct points } c_1, \ldots, c_r, d_1, \ldots d_r : d_i = M(c_i), 1 \leq i \leq r, \sum_{1 \leq k \leq r} \pi |c_k - d_k|^2 \leq 4c) \\
& \text{(using (19))} \\
\leq\ & {}^nC_r\, {}^{n-r}P_r Prob(\sum_{1 \leq k \leq r} \pi |c_k - d_k|^2 \leq 4c)) \\
& \text{(summing over all ways of choosing } r \text{ pairs)} \\
\leq\ & {}^nC_r\, {}^{n-r}P_r Prob(\sum_{1 \leq k \leq r} \lfloor \frac{\rho_k}{\delta} \rfloor^2 \leq \frac{4c}{\pi \delta^2}) \\
& \text{(where } \rho_k := |c_k - d_k| \text{ and } \delta > 0) \\
\leq\ & {}^nC_r\, {}^{n-r}P_r \sum_{(n_1, \ldots, n_r) \geq 0} Prob(\lfloor \frac{\rho_k}{\delta} \rfloor = n_k, \forall k, \text{ and } \sum_{1 \leq k \leq r} n_k^2 \leq \frac{4c}{\pi \delta^2}) \\
\leq\ & {}^nC_r\, {}^{n-r}P_r \sum_{\substack{(n_1, \ldots, n_r) \geq 0, \\ \sum_1^r n_k^2 \leq 4c/\pi\delta^2}} Prob(\lfloor \frac{\rho_k}{\delta} \rfloor = n_k, \forall k\ ) \\
\leq\ & {}^nC_r\, {}^{n-r}P_r \sum_{\substack{(n_1, \ldots, n_r) \geq 0, \\ \sum_1^r n_k^2 \leq 4c/\pi\delta^2}} (\prod_{k=1}^r \pi(2n_k + 1)\delta^2) \\
\leq\ & {}^nC_r\, {}^{n-r}P_r (\pi\delta^2)^r \sum_{\substack{(n_1, \ldots, n_r) \geq 0, \\ \sum_1^r n_k^2 \leq 4c/\pi\delta^2}} (\frac{r + \sum_1^r 2n_k}{r})^r \\
& \text{(using AM-GM ineq.)}
\end{aligned}$$

$$\leq \ {}^nC_r{}^{n-r}P_r(\pi\delta^2)^r \sum_{\substack{(n_1,\ldots,n_r)\geq 0,\\ \sum_1^r n_k^2 \leq 4c/\pi\delta^2}} (\frac{r+2\sqrt{\frac{4cr}{\pi\delta^2}}}{r})^r$$

(using Cauchy ineq., $\sum_1^r n_k \leq \sqrt{r\sum_1^r {n_k}^2}$ $\leq \sqrt{\frac{4cr}{\pi\delta^2}}$)

$$= \ {}^nC_r{}^{n-r}P_r(\pi\delta^2)^r(1+4\sqrt{\frac{c}{r\pi\delta^2}})^r \sum_{\substack{(n_1,\ldots,n_r)\geq 0,\\ \sum_1^r n_k^2 \leq 4c/\pi\delta^2}} 1$$

$$\leq \ {}^nC_r{}^{n-r}P_r(\pi\delta^2)^r(1+4\sqrt{\frac{c}{r\pi\delta^2}})^r 2^{r+\frac{4c}{\pi\delta^2}}. \qquad (21)$$

The last inequality follows from the fact that the number of non-negative integer solutions to $n_1+n_2+\ldots+n_r \leq c$, or equivalently to $n_1+n_2+\ldots+n_r+n_{r+1}=c$ is the coefficient of x^c in $(\frac{1}{1-x})^{r+1}$. This coefficient is ${}^{r+c}C_c$, which is less than 2^{r+c}.

To further simplify (21), note that for sufficiently large n and $0<r<n/2$,

$$\begin{aligned} & {}^nC_r{}^{n-r}P_r \\ = & \frac{n!}{r!(n-2r)!} \\ \leq & \frac{\sqrt{2\pi n}(\frac{n}{e})^{n+\frac{1}{12n}}}{\sqrt{2\pi r}(\frac{r}{e})^r(\frac{n-2r}{e})^{n-2r}} \text{ (using Stirling's approx.)} \\ < & \sqrt{n}(\frac{n}{e})^{r+\frac{1}{12n}}\frac{n^{n-r}}{r^r(n-2r)^{n-2r}} \\ < & (\frac{n}{e})^{r+1}(\frac{1}{\alpha^\alpha(1-2\alpha)^{1-2\alpha}})^n \text{ (where } \alpha := \tfrac{r}{n}) \\ < & (\frac{n}{e})^{r+1}(\frac{e^{2\alpha}}{\alpha^\alpha})^n \\ = & (\frac{n}{e})(\frac{n^2e}{r})^r. \end{aligned} \qquad (22)$$

The last inequality follows from the fact that for any $x>0$, $\frac{1}{x^x} \leq e^{1-x}$. To see this, note that $\ln t \leq t-1$ for $t>0$ and set $t=1/x$.

We now simplify (21) by setting $\delta = \sqrt{\frac{c}{r}}$ and using (22):

$$\begin{aligned} & Prob(\exists\ fn \text{ cells with total area} \leq c) \\ \leq & \ {}^nC_r{}^{n-r}P_r(\pi\delta^2)^r(1+4\sqrt{\frac{c}{r\pi\delta^2}})^r 2^{r+\frac{4c}{\pi\delta^2}} \\ < & (\frac{n}{e})(\frac{n^2e}{r})^r(\pi\frac{c}{r})^r(1+4\sqrt{\frac{1}{\pi}})^r 2^{r+\frac{4r}{\pi}} \\ = & (\frac{n}{e})(\pi ce(\frac{13}{f})^2(1+4\sqrt{\frac{1}{\pi}})2^{1+\frac{4}{\pi}})^r \\ = & (\frac{n}{e})(\frac{c_1}{f^2}c)^{\frac{fn}{13}} \\ \to & \ 0 \text{ as } n\to\infty \text{ (recalling } c < \tfrac{f^2}{c_1}). \end{aligned}$$

□

5.2 Transport Capacity of Random Wireless Networks with Infrastructure Support

We will now demonstrate how, by employing power control, a constant $\Theta(1)$ throughput can be supported for any fraction f, $0<f<1$, of nodes. We will assume that each source chooses a destination randomly. The routing strategy adopted by each node will be very simple. Each node always transmits directly to its closest base station, choosing its transmission range appropriately. The packet then traverses the infrastructure network to reach the base station closest to the destination node. Then the base station adjusts its transmission range, and transmits the packet over a wireless hop to its destination node.

We begin by showing which nodes can indeed achieve a constant throughput. Let X_i be any arbitrary base station, and let a_k be the generator of the Voronoi cell containing X_i. Let Δ be the guard zone parameter, as defined before for the Protocol Model.

DEFINITION 5.2. *Call a base station X_i* good *if the cell containing X_i, $V(a_k)$, is $(\pi(2+\Delta)^2|X_i-a_k|^2)$-fat. Call a Voronoi cell $V(a_k)$ itself* good *if it has at least one good base station, else call the cell itself* bad.

Next we show how the fraction of bad cells depends upon the number of base stations and the number of wireless nodes.

LEMMA 5.2. *Consider a random network with n wireless nodes and μn base stations. Suppose $0<f<1$ and $\mu > \frac{\ln\gamma_f}{c_f\beta_\Delta}$, where $\gamma_f := \frac{1}{(f)^f(1-f)^{1-f}}$, $c_f = \frac{f^2}{c_1}$ and $\beta_\Delta = \frac{0.41}{(2+\Delta)^2}$, with c_1 as defined in Lemma 5.1. Then*

$$\lim_{n\to\infty} Prob(\exists\ fn \text{ bad cells}) = 0. \qquad (23)$$

Proof: Consider a Voronoi cell, $V(a_k)$, which is bad. Let the fatness of $V(a_k)$ be t_k. Since the cell is bad, it is possible it has no base stations at all. Now suppose the cell has a base station X_i. Since $V(a_k)$ is $(\pi(2+\Delta)^2|X_i-a_k|^2)$-*thin*, we must have $\pi(2+\Delta)^2|X_i-a_k|^2 > t_k$, or equivalently, $\pi|X_i-a_k|^2 > \frac{t_k}{(2+\Delta)^2}$. Thus X_i does not lie inside a disk D_k centered at a_k with area $(\frac{t_k}{(2+\Delta)^2})$, or equivalently, disk D_k does not contain any base stations. With at least two cells in the domain D, the fatness t_k is at most the area of D and hence the area of D_k is at most 1/4 the area of D. Note that the fraction β of D_k that must lie inside D is maximized when D_k is the largest possible, and is given by $\beta \geq \cos^{-1}(\frac{1}{2\sqrt{4}})/\pi > 0.41$. Thus we have shown that at least an area $(\frac{\beta t_k}{(2+\Delta)^2}) = \beta_\Delta t_k$ is devoid of base stations. We now bound the probability that there exist at least fn bad cells:

$$\begin{aligned} & Prob(\exists\ fn \text{ bad cells}) \\ \leq & \ Prob(\exists\ fn \text{ bad cells whose total fatness} < c_f\ ) + \\ & Prob(\exists\ fn \text{ bad cells whose total fatness} \geq c_f\ ). \end{aligned}$$

Taking limits on both sides, the following holds:

$$\begin{aligned}
&\limsup_{n\to\infty} Prob(\exists\ fn \text{ bad cells}) \\
\leq\ &\limsup_{n\to\infty} Prob(\exists\ fn \text{ bad cells whose total fatness} \\
&< c_f\ ) + \limsup_{n\to\infty} Prob(\exists\ fn \text{ bad cells whose} \\
&\text{total fatness} \geq c_f\ ) \\
=\ &\limsup_{n\to\infty} Prob(\exists fn \text{ bad cells whose total fatness} \\
&\geq c_f) \text{ (using Lemma 5.1).} \qquad (24)
\end{aligned}$$

Now we upper bound the right hand side above by using the fact that for any bad cell, at least an area equal to a fraction β_Δ of its fatness is devoid of any base stations. Thus,

$$\begin{aligned}
&Prob(\exists\ fn \text{ bad cells whose total fatness} \geq c_f\ ) \\
\leq\ &\sum_{\{d_1,\ldots,d_{fn}\}} (Prob(\text{Cells } V(d_1),\ldots,V(d_{fn}) \text{ have total fatness } \geq c_f) \times Prob(\text{at least area } \beta_\Delta c_f \text{ of the } fn \text{ cells is devoid of base stations } \mid \text{Sum of the fatness of the } fn \text{ cells} \geq c_f)\ ) \\
\leq\ &(1-\beta_\Delta c_f)^{\mu n} \sum_{\{d_1,\ldots,d_{fn}\}} Prob(\text{Cells } V(d_1),\ldots,V(d_{fn}) \text{ have total fatness} \geq c_f) \\
\leq\ &{}^nC_{fn}(1-\beta_\Delta c_f)^{\mu n}.
\end{aligned}$$

We simplify this for large n by using the fact that $1-x \leq e^{-x}$ for all x, and

$$\begin{aligned}
&{}^nC_{fn} \\
=\ &\frac{n!}{(fn)!((1-f)n)!} \\
\leq\ &\frac{\sqrt{2\pi n}(\frac{n}{e})^{n+\frac{1}{12n}}}{\sqrt{2\pi fn}(\frac{fn}{e})^{fn}\sqrt{2\pi(1-f)n}(\frac{(1-f)n}{e})^{(1-f)n}} \\
\leq\ &(\frac{1}{(f)^f(1-f)^{1-f}})^n \text{ (for sufficiently large } n) \\
=\ &\gamma_f^n.
\end{aligned}$$

Thus,

$$\begin{aligned}
&Prob(\exists\ fn \text{ bad cells whose total fatness} \geq c_f\ ) \\
\leq\ &{}^nC_{fn}(1-\beta_\Delta c_f)^{\mu n} \\
\leq\ &\gamma_f^n e^{-\beta_\Delta c_f \mu n} \\
=\ &e^{n\ln(\gamma_f)-\beta_\Delta c_f \mu n} \\
\to\ &0 \text{ as } n \to 0.
\end{aligned}$$

Combining this with (24) yields the result. □

Lemma 5.2 gives us a lower bound on the ratio of the number of base stations to the number of wireless nodes, to guarantee that no more than a certain fraction f of the cells are bad. It can be shown that each value of $\mu > 0$ corresponds to a value of f, $0 < f < 1$. This is because the lower bound of μ, $\frac{\ln\gamma_f}{c_f\beta_f}$, is a monotonically decreasing function with limits ∞ and 0 respectively at $f = 0$ and $f = 1$. This implies that if the ratio of wireless nodes to base stations is bounded above, one always has some fraction of good cells in the network. As we will show, these good cells correspond to wireless nodes that can communicate with a base station without disturbing any other wireless node.

We are now ready to state our main result.

THEOREM 5.1. *For a random network with n wireless nodes and μn base stations, and where each node chooses a random destination, it is possible to provide $\Theta(\frac{1}{\log n})$ throughput to all the connections with high probability. In addition, simultaneously, it is possible to provide a $\Theta(1)$ throughput to a certain fraction f of wireless nodes. Formally, for each $\mu > 0$, there is an f, $0 < f < 1$, such that*

$$\lim_{n\to\infty} Prob(\exists fn \textit{ nodes that can achieve } \Theta(1) \textit{ throughput}) = 1. \qquad (25)$$

Proof: We begin by showing that any two good nodes, i.e., nodes with good Voronoi cells, can communicate with their respective closest base stations simultaneously without interfering with each other. Let the two good nodes be a_i and a_j and let their closest base stations be X_i and X_j. Let P be any point in $V(a_j)$. From the definition of a good node, we know that $V(a_i)$ is $(\pi(2+\Delta)^2|X_i - a_i|^2)$-fat. From the definition of $\eta - fat$, $|P - a_i| \geq (2+\Delta)|X_i - a_i|$. Using the triangle inequality, $|P - X_i| \geq |P - a_i| - |X_i - a_i| \geq (1+\Delta)|X_i - a_i|$. Since P is any point in $V(a_j)$, this shows that the transmission between X_i and a_i is not disturbed by the the transmission between X_j and a_j. This holds for any pair of nodes.

It is easy to see that any connection with both the source and destination as good nodes can be provided $\Theta(1)$ throughput. ¿From Lemma 5.2, we know that some constant fraction of nodes are good. Thus with high probability, some fn, $f > 0$, connections can be served $\Theta(1)$ throughput. It is easy to provide $\Theta(\frac{1}{\log n})$ throughput to all the other connections. Assuming a slotted operation, we can reserve all the even slots for the good connections selected above. In the odd slots, we can run another policy, like the one given in [6], that guarantees $\Theta(\frac{1}{\log n})$ throughput. □

6. CONCLUDING REMARKS

We have shown that for the Physical Model of communication, the best case transport capacity of an ad hoc wireless network grows asymptotically as $\Theta(\sqrt{n})$. The same bound holds even when the nodes are allowed to approach arbitrarily close to each other and even when the transmission rate is given by the Shannon's logarithmic function of the SINR. This closes the gap between the previously known upper bound of $\Theta(n^{\frac{\alpha-1}{\alpha}})$ and lower bound of $\Theta(\sqrt{n})$, where $\alpha > 2$ is the path loss exponent. We have also shown that the constraints of the Protocol Model are stricter than that of the Physical Model, thus allowing direct generalizations of lower bound constructions for the Protocol Model to the Physical Model.

For the case of hybrid wireless networks, we have shown how employing power control allows better scaling of network capacity. Specifically, one can guarantee a $\Theta(1)$ throughput to some $\Theta(n)$ nodes in the network. We show how the number of such nodes depends upon the number of base stations. Our result improves over previous results that establish a maximum throughput capacity of $\Theta(\frac{1}{\log n})$ per node in the absence of power control.

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