On the Path-Loss Attenuation Regime for Positive Cost and Linear Scaling of Transport Capacity in Wireless Networks

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***Abstract*— Wireless networks with a minimum inter-node separation distance are studied where the signal attenuation grows in magnitude as $\frac{1}{\rho^{\delta}}$ with distance ρ. Two performance measures of wireless networks are analyzed. The transport capacity is the supremum of the total distance-rate products that can be supported by the network. The energy cost of information transport is the infimum of the ratio of the transmission energies used by all the nodes to the number of bit-meters of information thereby transported.**

If the phases of the attenuations between node pairs are uniformly and independently distributed, it is shown that the expected transport capacity is upper bounded by a multiple of the total of the transmission powers of all the nodes, whenever $\delta > 2$ for two-dimensional networks or $\delta > \frac{5}{4}$ for one-dimensional networks, even if all the nodes have full knowledge of all the phases, i.e., full channel state information. If all nodes have an individual power constraint, the expected transport capacity grows at most linearly in the number of nodes due to the linear growth of the total power. This establishes the best case order of expected transport capacity for these ranges of path-loss exponents since linear scaling is also feasible.

If the phases of the attenuations are arbitrary, it is shown that the transport capacity is upper bounded by a multiple of the total transmission power whenever $\delta > \frac{5}{2}$ for two-dimensional networks or $\delta > \frac{3}{2}$ for one-dimensional networks, even if all the nodes have full channel state information. This shows that there is indeed a positive energy cost which is no less than the reciprocal of the above multiplicative constant. It narrows the transition regime where the behavior is still open, since it is known that when $\delta < \frac{3}{2}$ for two-dimensional networks, or $\delta < 1$ for one-dimensional networks, the transport capacity cannot generally be bounded by any multiple of the total transmit power.

***Index Terms*— Ad hoc networks, capacity of wireless networks, cut-set bound, max-flow min-cut bound, multi-user information theory, network information theory, scaling laws, transport capacity, wireless networks.**

I. Introduction

Wireless networks formed by nodes with radios are a subject of much topical interest, and may be at the cusp of a take-off. They are of interest not only in ad hoc wireless networks [1], but also in mesh networks [2], [3], sensor networks [4], [5], [6], and the emerging field of control over wireless networks [7]. It is of importance to understand what such networks are capable of supporting, and how to operate them to maximize their capabilities.

For the communication functionality, a fundamental question is: How much information can a wireless network transport? To answer this question, one naturally turns to the field of Information Theory. However, *network* information theory for communication channels with multiple users is an area where even several simple scenarios, such as the relay channel and the interference channel, have not been completely solved [8, Chapter 14], even though there has been success with respect to the multiple access channel and the Gaussian broadcast channel.

In a previous paper [9], the capacity of wireless networks was studied under technological models where interference gives rise to collisions. It was shown that the capability of a wireless network manifests itself not only in the information transmission *rate*, but also in the information transmission *distance*. To reflect this, the concept of *transport capacity* was introduced to account for the total rate-distance product (in the unit "bit-meters/time unit") that a wireless network can support. One key result obtained was that the transport capacity of a wireless network grows at most like the square root of the product of the area of the network and the number of the nodes. Another was that if the node locations are random, and every node chooses a random destination for its originating traffic, then, as the number n of nodes increases, there is a sharp cutoff of $\Theta(\frac{1}{\sqrt{n \log n}})$ for the uniform rate that can be supported for every such source-destination pair. The scaling laws obtained in [9] were however not conclusive due to the restrictive models studied, which do not cover technologies such as successive interference cancellation (perhaps better called "subtraction" rather than "cancellation" due to its liability for confusion with the next possibility) or active interference cancellation, or operational strategies such as amplifying and forwarding without decoding, etc. To an information theorist, the ultimate goal is to find out what is possible or impossible, without such technological presumptions.

In a subsequent paper [10], general wireless networks were therefore studied in an information theoretic setting. Since "distance" plays such a crucial role, as evidenced by the conservation laws for the transport capacity alluded to

Manuscript received March 6, 2005; revised January 24, 2006. This work was supported in part by NSF under Contracts NSF-ANI-02-21357 and CCR-0325716, USARO under Contracts DAAD19-00-1-0466 and DAAD19-01010-465, DARPA/AFOSR under Contract F49620-02-1-0325, AFOSR under Contract F49620-02-1-0217, and DARPA under Contacts N00014-0-1-1-0576 and F33615-0-1-C-1905. Part of the work of L.-L. Xie was conducted while he was visiting the University of Illinois at Urbana-Champaign.

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above, it was incorporated into the model not only through making explicit the distances between nodes, but also through explicitly modeling the attenuation of signals with distance ρ by the factor $\frac{e^{-\gamma\rho}}{\rho^{\delta}}$. To reflect antenna considerations, a minimum separation distance between nodes was presumed, which also avoids singularities at $\rho = 0$. A fundamental result established in [10] is that the transport capacity is always upper bounded by a multiple of the total power used by the transmissions of all the nodes in the network, provided that the signals are attenuated sufficiently with distance. This multiple thus corresponds to the maximum bit-meters of transport that a network can deliver per unit energy consumed by transmissions. It was shown that either $\gamma > 0$, or $\gamma = 0$ but $\delta > 3$, was sufficient for the existence of such a energy cost per unit transport, while $\gamma > 0$, or $\gamma = 0$ but $\delta > 2$, was sufficient for linear networks where the nodes are arranged along a line. On the other hand, counterexamples were also provided of multiple relay networks to show that if $\gamma = 0$ but $\delta < \frac{3}{2}$ for two-dimensional networks, or $\gamma = 0$ but $\delta < 1$ for one-dimensional networks, then the transport capacity can indeed be unbounded even with bounded total transmission power.

For wireless networks where each node has the same constraint on its transmission power, the above result immediately establishes a linear scaling law for the transport capacity, since the total transmission power itself grows linearly in the number of nodes. This is a slight sharpening of, but in essential conformity with, the $O\left(\sqrt{An}\right)$ scaling law established in [9], since the area of the domain grows at least like n with a minimum inter-node spacing. Since linear scaling is in fact achievable, as constructively shown in [9], and that too using only simple decode-and-forward multiple hopping where at each hop all concurrent interference is treated as noise, the optimality of the order of the best case transport capacity is thus established for the range of attenuations where this linear scaling is established. Note that this also proves that the above architecture for information transport is optimal to within a constant factor.

Thus interest centers on determining precisely for what range of path loss exponents δ (with $\gamma = 0$) linear scaling of transport capacity can indeed be established. From the aforementioned results of [10], there is a gap $\frac{3}{2} \leq \delta \leq 3$ for two-dimensional networks, and $1 \leq \delta \leq 2$ for one-dimensional networks, where the scaling law behavior is unknown. In a subsequent work [11], an improvement was made, and it was proved that $\delta > \frac{5}{2}$ for two-dimensional networks and $\delta > \frac{3}{2}$ for one-dimensional networks, were also sufficient for linear scaling to hold.

Instead of transport capacity, the average rate per communication pair was examined in [12]. It was shown that in a network with sufficiently many randomly chosen communication pairs, this average rate tends to zero as the number of nodes in the network grows to infinity. For this result, the required attenuation exponent δ is much smaller ($\delta > 1$ for two-dimensional networks and $\delta > \frac{1}{2}$ for one-dimensional networks) compared with that needed for linear scaling of the transport capacity.

In all these works [10], [11], [12], the information-theoretical tool used to prove the upper bounds is the cut-set bound, which is also known as the max-flow min-cut bound; see [8, Section 14.10] for a general formulation in terms of mutual informations. For the specific application to wireless networks, a formula in terms of powers was presented in [10].

Essentially, the cut-set bound is an application of Fano's Inequality to the network scenario. It is known that Fano's Inequality provides a tight upper bound on the rate achievable from one source to one destination. For a network with multiple nodes, the idea is to dissect it into two sets, with one regarded as the virtual "source" and the other as the virtual "destination." Then by Fano's Inequality, one can bound the total rates achievable from the nodes in the "source" set to the nodes in the "destination" set. However, this bound is no longer tight, unless all the nodes in the "source" set can cooperate in the encoding, and also all the nodes in the "destination" set can cooperate in the decoding, which are both generally not feasible.

However, up to now, the cut-set bound appears to be the only general tool that can be used for establishing upper bounds on the capacity of networks. Nevertheless, one can obtain sharper bounds and thus tighter results by considering multiple cuts through the network *simultaneously*. Since multiple single cuts considered separately may not be all maximized *simultaneously* with the same distribution of the inputs, such a cut-set bound with multiple cuts is generally tighter than a simple combination of multiple single cut-set bounds. In this paper, we will employ such a cut-set bound with multiple cuts to get tighter bounds on transport capacity. Besides, we will prove that for Gaussian wireless networks, a joint Gaussian distribution of the inputs achieves the maximum for the cut-set bound with multiple cuts.

Actually, a two-cut version of the cut-set bound with multiple cuts has appeared in [13], where it was used to prove the converse for the capacity of physically degraded Gaussian relay channels. Later on, another two-cut version was applied in [14, Chapter 2] to get tighter upper bounds on the capacity of Gaussian parallel relay channels.

In this paper, we will first present a general formula for the cut-set bound with multiple cuts in Section III, where, more importantly for the treatment of Gaussian wireless networks, we will prove the optimality of a joint Gaussian distribution of the inputs. Applications of the cut-set bound with multiple cuts to one-dimensional and two-dimensional networks are made in Sections IV and V respectively. Two cases are treated. Assuming random phases of the signal attenuations (but assuming that they are known to all the transmitters and receivers, i.e., full channel state information available at all nodes), we prove that $\delta > 2$ for two-dimensional networks, and $\delta > \frac{5}{4}$ for one-dimensional networks, are sufficient for establishing that the expected transport capacity is upper bounded by a multiple of the total of the transmissions of all the nodes. If nodes are each subject to an individual power constraint, then it follows that the expected transport capacity scales at best linearly in the number of nodes. This is sharp in the best case since linear scaling is indeed feasible for the transport capacity. In the case that the phases are arbitrary, then uniformly for all realizations of the phases, the transport capacity is upper bounded by a multiple of the total of the transmission powers if $\delta > \frac{5}{2}$ for two-dimensional networks or $\delta > \frac{3}{2}$ for one-dimensional networks, even if all nodes have full information on the states of all the channels. Thus there is indeed a minimal positive energy cost per bit-meter of information transport. This narrows the attenuation regime where the behavior is still unknown, to the interval of path loss exponents $\frac{3}{2} \leq \delta \leq \frac{5}{2}$ for two-dimensional networks, and to $1 \leq \delta \leq \frac{3}{2}$ for one-dimensional networks, since for values of δ below these ranges it has been shown that there are networks whose infimum of energy costs per bit-meter of transport is indeed zero. We also show that unless one can improve on the bound following from the multiple cuts, one cannot establish linear scaling in this unknown region.

II. Model and Definitions

Consider a wireless network consisting of n nodes $\mathcal{N} = \{1, 2, \dots, n\}$. Let $X_i(t) \in \mathbb{C}^1$ or $Y_i(t) \in \mathbb{C}^1$ respectively denote the signal sent or received by Node $i \in \mathcal{N}$ at the time instant $t = 1, 2, \dots$. Each node receives a measurement that is an attenuated and superposed combination of all the other transmissions and white Gaussian noise:

$$Y_j(t) = \sum_{\substack{i \in \mathcal{N} \\ i \neq j}} g_{i,j} X_i(t) + Z_j(t), \qquad \forall\, j \in \mathcal{N}, \quad t = 1, 2, \dots \tag{1}$$

Here $\{g_{i,j} \in \mathbb{C}^1 : i \neq j\}$ denote the signal attenuation gains, and $Z_i(t)$ are zero-mean complex Gaussian noise with independent, equal variance real and imaginary parts. For each i, $\{Z_i(t), t = 1, 2, \dots\}$ are i.i.d, and for different i or t, $\{Z_i(t)\}$ are independent of each other.

It is convenient to define the gain matrix

$$G := \begin{pmatrix} g_{1,1} & g_{1,2} & \cdots & g_{1,n} \\ g_{2,1} & g_{2,2} & \cdots & g_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ g_{n,1} & g_{n,2} & \cdots & g_{n,n} \end{pmatrix} \tag{2}$$

with $g_{i,i} := 0$ for all $i = 1, \dots, n$, and to define vectors $\underline{X} = (X_1, X_2, \dots, X_n)^T$, $\underline{Y} = (Y_1, Y_2, \dots, Y_n)^T$ and $\underline{Z} = (Z_1, Z_2, \dots, Z_n)^T$. Then (1) can be put into a compact form:

$$\underline{Y}(t) = G^T \underline{X}(t) + \underline{Z}(t), \qquad t = 1, 2, \dots \tag{3}$$

Next, we recall the definition of the transport capacity of a wireless network [10]. For this, we gather together the by now pro-forma standard definitions of information theory including codes with power constraint, the probability of error, achievable rates, etc., for wireless networks.

Definition 2.1: Consider a wireless network of n nodes. Let $\{(s_\ell, d_\ell),\ \ell = 1, \dots, n(n-1)\}$, be a listing of the $n(n-1)$ possible source-destination pairs. Then a $\big((2^{TR_1}, \dots, 2^{TR_{n(n-1)}}), T, P_e^{(T)}\big)$, code with power constraints $\{P_i,\, i \in \mathcal{N}\}$, consists of the following:

1) Independent random variables $\{W_\ell : 1 \leq \ell \leq n(n-1)\}$, with $P(W_\ell = k_\ell) = \frac{1}{2^{TR_\ell}}$ for every $k_\ell \in \{1, 2, \dots, 2^{TR_\ell}\}$. Let $\overline{W}_i := \{W_\ell : s_\ell = i\}$ and $\overline{R}_i := \sum_{\{\ell :\, s_\ell = i\}} R_\ell$. (Note that $\overline{W}_j = \emptyset$ and $\overline{R}_j = 0$ if no traffic originates at node j).
2) Functions $f_{i,t} : \mathbb{C}^{t-1} \times \{1, 2, \dots, 2^{T\overline{R}_i}\} \to \mathbb{C}^1, t = 1, 2, \dots, T$, for node $i = 1, 2, \dots, n$, such that
$$X_i(t) = f_{i,t}\big(Y_i(1), \dots, Y_i(t-1), \overline{W}_i\big), \quad t = 1, 2, \dots, T;$$
with
$$X_j(1) = 0 \text{ for nodes with } \overline{R}_j = 0,$$
such that the following power constraints hold:
$$\frac{1}{T} \sum_{t=1}^{T} \|X_i(t)\|^2 \quad \leq \quad P_i\,, \quad \text{a.s.,} \quad \text{for } i \in \mathcal{N}. \tag{4}$$
3) Decoding functions $g_\ell : \mathbb{C}^T \times \{1, 2, \dots, 2^{T\overline{R}_{d_\ell}}\} \to \{1, 2, \dots, 2^{TR_\ell}\}$ for $\ell = 1, 2, \dots, n(n-1)$.
4) The average probability of error:
$$P_e^{(T)} := \text{Prob}\big((\hat{W}_1, \hat{W}_2, \dots, \hat{W}_m) \neq (W_1, W_2, \dots, W_m)\big) \tag{5}$$
where $\hat{W}_\ell := g_\ell\big(Y_{d_\ell}^T, \overline{W}_{d_\ell}\big)$, with $Y_{d_\ell}^T := \big(Y_{d_\ell}(1), Y_{d_\ell}(2), \dots, Y_{d_\ell}(T)\big)$.

Definition 2.2: A rate vector $(R_1, \dots, R_{n(n-1)})$ is said to be *achievable* for the $n(n-1)$ source-destination pairs $\{(s_\ell, d_\ell), \ell = 1, \dots, n(n-1)\}$, with the power constraints $\{P_i,\, i \in \mathcal{N}\}$, if there exists a sequence of $\big((2^{TR_1}, \dots, 2^{TR_{n(n-1)}}), T, P_e^{(T)}\big)$ codes with power constraints $\{P_i,\, i \in \mathcal{N}\}$ such that $P_e^{(T)} \to 0$ as $T \to \infty$.

The above definitions are presented in the context of the separate power constraints $\{P_i,\, i \in \mathcal{N}\}$ for the n nodes. However, if a total power constraint P_{total}, or a common individual power constraint P_{ind}, is imposed, then one simply needs to replace the constraints (4) by

$$\frac{1}{T} \sum_{t=1}^{T} \sum_{i \in \mathcal{N}} \|X_i(t)\|^2 \quad \leq \quad P_{\text{total}}\,, \quad \text{a.s.} \tag{6}$$

or

$$\frac{1}{T} \sum_{t=1}^{T} \|X_i(t)\|^2 \quad \leq \quad P_{\text{ind}}, \quad \text{a.s.,} \quad \text{for } i \in \mathcal{N}, \tag{7}$$

and correspondingly modify the rest of the definitions.

Above, we have not considered node locations, or even distances between nodes. Let ρ_ℓ denote the distance between source s_ℓ and destination d_ℓ, in the ℓ-th source-destination pair $(s_\ell,\, d_\ell)$.

Definition 2.3: The *transport capacity* of a wireless network is defined as

$$C_{\text{T}} \quad := \sup_{(R_1, \dots, R_{n(n-1)}) \text{ achievable}} \sum_{\ell=1}^{n(n-1)} R_\ell \cdot \rho_\ell.$$

III. A Cut-Set Bound with Multiple Cuts

Let S be any subset of the nodes $\mathcal{N}$ in the network, and denote its complement by $S^c = \mathcal{N} - S$. Let $R^{(S)}$ be the summation of all the achievable rates with the source in S and the destination in S^c. Then by [8, Theorem 14.10.1], we have

$$R^{(S)} \le \max_{p(x_1,\ldots,x_n)} I(X^{(S)}; Y^{(S^c)} | X^{(S^c)}) \tag{8}$$

where $X^{(S)} := \{X_i,\, i \in S\}$, and $Y^{(S^c)}$, $X^{(S^c)}$ are similarly defined.

One can also consider multiple cuts of the network simultaneously, to obtain the following corollary.

Corollary 3.1: Considering multiple subsets $S_k \subset \mathcal{N}$, $k = 1, 2 \ldots, K$, simultaneously, we have the following bound:

$$\sum_{k=1}^{K} \alpha_k R^{(S_k)} \le \max_{p(x_1,\ldots,x_n)} \sum_{k=1}^{K} \alpha_k I(X^{(S_k)}; Y^{(S_k^c)} | X^{(S_k^c)}) \tag{9}$$

where $\alpha_k \ge 0$, $k = 1, \ldots, K$ are arbitrary weights.

Proof: From (14.313)-(14.330) in [8, p.446], for any S_k, $k = 1, 2 \ldots, K$, we have

$$R^{(S_k)} \le I(X_Q^{(S_k)}; Y_Q^{(S_k^c)} | X_Q^{(S_k^c)}) + \epsilon_T^{(k)} \tag{10}$$

where Q is a random variable uniformly distributed on the set $\{1, 2, \ldots, T\}$, and $\epsilon_T^{(k)} \to 0$ as $T \to \infty$. (Note that Q only depends on T and is independent of k.)

A weighted summation of (10) gives

$$\sum_{k=1}^{K} \alpha_k R^{(S_k)} \le \sum_{k=1}^{K} \alpha_k I(X_Q^{(S_k)}; Y_Q^{(S_k^c)} | X_Q^{(S_k^c)}) + \sum_{k=1}^{K} \alpha_k \epsilon_T^{(k)}$$

which leads to (9) immediately by letting $T \to \infty$. □

Remark 3.1: The bound (9) is in general tighter than applying the single cut bound (8) K times on the subsets S_k, $k = 1, 2 \ldots, K$, which leads to

$$\sum_{k=1}^{K} \alpha_k R^{(S_k)} \le \sum_{k=1}^{K} \alpha_k \max_{p(x_1,\ldots,x_n)} I(X^{(S_k)}; Y^{(S_k^c)} | X^{(S_k^c)})$$

Now, we turn to the wireless network with power constraints defined in last section. Consider the following subsets:

$$S_i := \{1, \ldots, i\} \subset \mathcal{N}, \qquad \forall\, i = 1, 2, \ldots, n-1, \tag{11}$$

and we have the following theorem.

Theorem 3.1: Considering the $n-1$ subsets S_i defined in (11) simultaneously, we have the following bound:

$$\sum_{i=1}^{n-1} \alpha_i R^{(S_i)} \le \max_{\left\{P_{k\ell} \ge 0:\, \sum_{\ell=k}^{n-1} P_{k\ell} \le P_k\right\}} \max_{\{\phi_{k\ell} \in [0, 2\pi)\}} \sum_{i=1}^{n-1} \alpha_i \sum_{j=i+1}^{n} \log\left(1 + \frac{\sum_{\ell=1}^{i} \left\| \sum_{k=1}^{\ell} g_{k,j} \sqrt{P_{k\ell}}\; e^{\mathbf{i}\phi_{k\ell}} \right\|^2}{\sigma_j^2}\right) \tag{12}$$

where $\alpha_i \ge 0$, for $i = 1, \ldots, n-1$, are arbitrary weights, P_k is the power constraint of node $k \in \mathcal{N}$, and $\sigma_j^2 = \mathbb{E}\|Z_j(1)\|^2$ for j=2,…,n.

Proof: An application of Corollary 3.1 gives

$$\sum_{i=1}^{n-1} \alpha_i R^{(S_i)} \le \max_{p(x_1,\ldots,x_n)} \sum_{i=1}^{n-1} \alpha_i I(X_1, \ldots, X_i; Y_{i+1}, \ldots, Y_n | X_{i+1}, \ldots, X_n). \tag{13}$$

By the model (1), for any $i = 1, \ldots, n-1$,

$$\begin{aligned}
& I(X_1, \ldots, X_i; Y_{i+1}, \ldots, Y_n | X_{i+1}, \ldots, X_n) \\
&= h(Y_{i+1}, \ldots, Y_n | X_{i+1}, \ldots, X_n) \\
&\quad - h(Y_{i+1}, \ldots, Y_n | X_1, \ldots, X_n) \\
&= h\left(\textstyle\sum_{k=1}^{i} g_{k,i+1} X_k + Z_{i+1}, \ldots, \sum_{k=1}^{i} g_{k,n} X_k + Z_n \,\middle|\, X_{i+1}, \ldots, X_n\right) \\
&\quad - h(Z_{i+1}, \ldots, Z_n) \\
&\le \textstyle\sum_{j=i+1}^{n} h\left(\sum_{k=1}^{i} g_{k,j} X_k + Z_j \,\middle|\, X_{i+1}, \ldots, X_n\right) \\
&\quad - \textstyle\sum_{j=i+1}^{n} h(Z_j) \\
&\le \textstyle\sum_{j=i+1}^{n} h\left(\sum_{k=1}^{i} g_{k,j} X_k + Z_j \,\middle|\, X_{i+1}, \ldots, X_{n-1}\right) \\
&\quad - \textstyle\sum_{j=i+1}^{n} h(Z_j)
\end{aligned}$$

where in the last inequality, "=" holds if X_n is independent of all X_i, $i = 1, \ldots, n-1$. Therefore, by (13),

$$\sum_{i=1}^{n-1} \alpha_i R^{(S_i)} \le \max_{p(x_1,\ldots,x_{n-1})} \sum_{i=1}^{n-1} \alpha_i \sum_{j=i+1}^{n} h\left(\sum_{k=1}^{i} g_{k,j} X_k + Z_j \,\middle|\, X_{i+1}, \ldots, X_{n-1}\right) - \sum_{i=1}^{n-1} \alpha_i \sum_{j=i+1}^{n} h(Z_j). \tag{14}$$

Now, consider any $p(x_1, \ldots, x_{n-1})$ with $\mathbb{E}\|X_i\|^2 > 0$, $i = 1, \ldots, n-1$. Define the correlation matrix of the vector $(X_1, \ldots, X_{n-1})^T$ by

$$\begin{aligned}
\Gamma &:= \begin{pmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1,n-1} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{n-1,1} & \gamma_{n-1,2} & \cdots & \gamma_{n-1,n-1} \end{pmatrix} \\
&:= \mathbb{E} \begin{pmatrix} \frac{X_1}{\sqrt{\mathbb{E}\|X_1\|^2}} \\ \vdots \\ \frac{X_{n-1}}{\sqrt{\mathbb{E}\|X_{n-1}\|^2}} \end{pmatrix} \begin{pmatrix} \frac{X_1}{\sqrt{\mathbb{E}\|X_1\|^2}} \\ \vdots \\ \frac{X_{n-1}}{\sqrt{\mathbb{E}\|X_{n-1}\|^2}} \end{pmatrix}^\dagger
\end{aligned} \tag{15}$$

where $(\cdot)^\dagger$ denotes the conjugate transpose. Note that Γ is different from the covariance matrix since X_i's may not be zero-mean.

For any two complex random variables $(X, Y) \in \mathbb{C}^2$, let

$$\begin{aligned}
\mathrm{Var}(Y|X) &:= \mathbb{E}[\|Y - \mathbb{E}(Y|X)\|^2 | X] \quad \text{and} \\
\gamma_{XY} &:= \frac{\mathbb{E} X Y^*}{\sqrt{\mathbb{E}\|X\|^2 \mathbb{E}\|Y\|^2}}
\end{aligned}$$

where $(\cdot)^*$ denotes the conjugate. From the Appendix,

$$\mathbb{E}[\mathrm{Var}(Y|X)] \le (1 - \|\gamma_{XY}\|^2) \mathbb{E}\|Y\|^2 \tag{16}$$

with "=" holding in (16) if (X,Y) is circularly symmetric complex Gaussian distributed with zero mean (see [15, Section 2] for the definition). Hence, we have

$$\begin{aligned} h(Y|X) &\le \mathbb{E}\left\{\log[\pi e \text{Var}(Y|X)]\right\} && (17)\\ &\le \log\left\{\pi e \mathbb{E}[\text{Var}(Y|X)]\right\} && (18)\\ &\le \log\left[\pi e(1-\|\gamma_{XY}\|^2)\mathbb{E}\|Y\|^2\right] && (19)\end{aligned}$$

where (18) follows from the Jensen's Inequality, and "=" holds in both the inequalities (17) and (19) if (X,Y) is circularly symmetric complex Gaussian distributed with zero mean.

Therefore, for any $1 \le i < j \le n$,

$$\begin{aligned} & h\left(\sum_{k=1}^{i} g_{k,j}X_k + Z_j \,\middle|\, X_{i+1},\ldots,X_{n-1}\right)\\ \le & \min_{\substack{\beta_q^{(i,j)}\in\mathbb{C}^1\\ q=i+1,\ldots,n-1}} h\left(\sum_{k=1}^{i} g_{k,j}X_k + Z_j \,\middle|\, \sum_{q=i+1}^{n-1}\beta_q^{(i,j)}X_q\right)\\ \le & \min_{\substack{\beta_q^{(i,j)}\in\mathbb{C}^1\\ q=i+1,\ldots,n-1}} \log \qquad (20)\\ & \left[\pi e\left(1-\|\gamma_{(i,j,\beta_{i+1}^{(i,j)},\ldots,\beta_{n-1}^{(i,j)})}\|^2\right)\mathbb{E}\left\|\sum_{k=1}^{i} g_{k,j}X_k+Z_j\right\|^2\right]\end{aligned}$$

where

$$\gamma_{(i,j,\beta_{i+1}^{(i,j)},\ldots,\beta_{n-1}^{(i,j)})} := \frac{\mathbb{E}\left(\sum_{k=1}^{i} g_{k,j}X_k+Z_j\right)\left(\sum_{q=i+1}^{n-1}\beta_q^{(i,j)}X_q\right)^*}{\sqrt{\mathbb{E}\left\|\sum_{k=1}^{i} g_{k,j}X_k+Z_j\right\|^2\mathbb{E}\left\|\sum_{q=i+1}^{n-1}\beta_q^{(i,j)}X_q\right\|^2}}.$$

Let $P'_i := \mathbb{E}\|X_i\|^2$. Define vectors $\underline{P}' := (P'_1,\ldots,P'_{n-1})^T$ and $\underline{\sigma}^2 := (\sigma_2^2,\ldots,\sigma_n^2)^T$. By $\underline{P}'>0$, we mean that all $P'_i > 0$, $i=1,\ldots,n-1$. It is easy to see that in (20), the bound minimized over $\left\{\beta_q^{(i,j)}\in\mathbb{R}^1,\, q=i+1,\ldots,n-1\right\}$ only depends on i, j, $\underline{P}'$, $\underline{\sigma}^2$, $G=\{g_{ij}\}$ and $\Gamma=\{\gamma_{ij}\}$, and thus can be denoted by

$$B_{i,j}\left(\underline{P}',\underline{\sigma}^2,G,\Gamma\right). \qquad (21)$$

Therefore, by (14),

$$\begin{aligned} & \sum_{i=1}^{n-1}\alpha_i R^{(S_i)} + \sum_{i=1}^{n-1}\alpha_i\sum_{j=i+1}^{n} h(Z_j)\\ & \le \sup_{\substack{p(x_1,\ldots,x_{n-1})\\ \underline{P}'>0}} \sum_{i=1}^{n-1}\alpha_i\sum_{j=i+1}^{n} B_{i,j}\left(\underline{P}',\underline{\sigma}^2,G,\Gamma\right)\\ & = \sup_{\underline{P}'>0}\max_{\Gamma}\sum_{i=1}^{n-1}\alpha_i\sum_{j=i+1}^{n} B_{i,j}\left(\underline{P}',\underline{\sigma}^2,G,\Gamma\right) \qquad (22)\\ & = \sup_{\underline{P}'>0}\sup_{\Gamma>0}\sum_{i=1}^{n-1}\alpha_i\sum_{j=i+1}^{n} B_{i,j}\left(\underline{P}',\underline{\sigma}^2,G,\Gamma\right) \qquad (23)\end{aligned}$$

where (22) follows since the bound only depends on $p(x_1,\ldots,x_{n-1})$ via $\underline{P}'$ and Γ, and, noting that Γ by the definition (15) is nonnegative definite, (23) follows by restricting Γ to be positive definite.

Now, for any given $\Gamma > 0$ and $\underline{P}' > 0$, we specially construct $p(x_1,\ldots,x_{n-1})$ as follows. Let $\xi_i \in \mathbb{C}^1$, $i = 1,\ldots,n-1$ be i.i.d. zero-mean complex Gaussian distributed with independent real and imaginary parts of the same variance $\frac{1}{2}$. Let

$$X_i = \sum_{k=i}^{n-1}\lambda_{ik}\xi_k\sqrt{P'_i}, \qquad \forall i=1,\ldots,n-1,$$

where $\{\lambda_{ik}\in\mathbb{C}^1,\, 1\le i\le k\le n-1\}$ satisfy

$$\Gamma = \begin{pmatrix} \lambda_{11} & 0 & \cdots & 0\\ \lambda_{12} & \lambda_{22} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ \lambda_{1,n-1} & \lambda_{2,n-1} & \cdots & \lambda_{n-1,n-1}\end{pmatrix}^{\dagger} \times \begin{pmatrix} \lambda_{11} & 0 & \cdots & 0\\ \lambda_{12} & \lambda_{22} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ \lambda_{1,n-1} & \lambda_{2,n-1} & \cdots & \lambda_{n-1,n-1}\end{pmatrix} \qquad (24)$$

with $\lambda_{ii}\neq 0$ for $i=1,\ldots,n-1$. The decomposition (24) is possible since Γ is a positive definite matrix.

For this special construction of $\{X_i\}$, we can choose $\hat{\beta}_q^{(i,j)}\in\mathbb{C}^1$, $q=i+1,\ldots,n-1$ such that

$$\sum_{k=1}^{i} g_{k,j}X_k - \sum_{q=i+1}^{n-1}\hat{\beta}_q^{(i,j)}X_q = \sum_{k=1}^{i} g_{k,j}\sum_{\ell=k}^{i}\lambda_{k\ell}\xi_\ell\sqrt{P'_k}$$

which can be easily checked noting that $\lambda_{ii}\neq 0$ for any $i=1,\ldots,n-1$. Hence, for the terms in (20), we have

$$\begin{aligned} & h\left(\sum_{k=1}^{i} g_{k,j}X_k + Z_j \,\middle|\, \sum_{q=i+1}^{n-1}\hat{\beta}_q^{(i,j)}X_q\right)\\ & \qquad = h\left(\sum_{k=1}^{i} g_{k,j}\sum_{\ell=k}^{i}\lambda_{k\ell}\xi_\ell\sqrt{P'_k}+Z_j\right) \qquad (25)\end{aligned}$$

and

$$\begin{aligned} & h\left(\sum_{k=1}^{i} g_{k,j}X_k + Z_j \,\middle|\, \sum_{q=i+1}^{n-1}\hat{\beta}_q^{(i,j)}X_q\right) = \qquad (26)\\ & \log\left[\pi e\left(1-\|\gamma_{(i,j,\hat{\beta}_{i+1}^{(i,j)},\ldots,\hat{\beta}_{n-1}^{(i,j)})}\|^2\right)\mathbb{E}\left\|\sum_{k=1}^{i} g_{k,j}X_k+Z_j\right\|^2\right]\end{aligned}$$

where the equality in (26) holds since "=" holds in all the three inequalities (17), (18) and (19) with

$$Y=\sum_{k=1}^{i} g_{k,j}X_k+Z_j, \qquad X=\sum_{q=i+1}^{n-1}\hat{\beta}_q^{(i,j)}X_q.$$

Therefore, by (20), (21), (25) and (26),

$$B_{i,j}\left(\underline{P}',\underline{\sigma}^2,G,\Gamma\right) \le h\left(\sum_{k=1}^{i} g_{k,j}\sum_{\ell=k}^{i}\lambda_{k\ell}\xi_\ell\sqrt{P'_k}+Z_j\right).$$

Hence, by (23), we have

$$\sum_{i=1}^{n-1}\alpha_i R^{(S_i)} \le \sup_{\underline{P}'>0}\sup_{\Gamma>0}\sum_{i=1}^{n-1}\alpha_i\sum_{j=i+1}^{n} h\left(\sum_{k=1}^{i} g_{k,j}\sum_{\ell=k}^{i}\lambda_{k\ell}\xi_\ell\sqrt{P'_k}+Z_j\right)-\sum_{i=1}^{n-1}\alpha_i\sum_{j=i+1}^{n}h(Z_j)$$
$$=\sup_{\underline{P}'>0}\sup_{\Gamma>0}\sum_{i=1}^{n-1}\alpha_i\sum_{j=i+1}^{n}\log\frac{\mathbb{E}\left\|\sum_{k=1}^{i}g_{k,j}\sum_{\ell=k}^{i}\lambda_{k\ell}\xi_\ell\sqrt{P'_k}+Z_i\right\|^2}{\mathbb{E}\|Z_j\|^2}$$
$$=\sup_{\underline{P}'>0}\sup_{\Gamma>0}\sum_{i=1}^{n-1}\alpha_i\sum_{j=i+1}^{n}\log\left(1+\frac{\sum_{\ell=1}^{i}\left\|\sum_{k=1}^{\ell}g_{k,j}\lambda_{k\ell}\sqrt{P'_k}\right\|^2}{\sigma_j^2}\right). \tag{27}$$

By (15) and (24), we have that for any Γ,

$$\sum_{\ell=k}^{n-1}\|\lambda_{k\ell}\|^2=\gamma_{kk}=1.$$

Also note that the power constraints ensure that

$$P'_k\le P_k+\epsilon_k,\quad \forall\,\epsilon_k>0,\quad \text{for } k=1,\dots,n-1.$$

Hence, by (27), we finally have

$$\sum_{i=1}^{n-1}\alpha_i R^{(S_i)}\le \lim_{\{\epsilon_k\to 0\}}\ \sup_{\{0<P'_k\le P_k+\epsilon_k\}}\ \max_{\left\{\sum_{\ell=k}^{n-1}\|\lambda_{k\ell}\|^2=1\right\}}\sum_{i=1}^{n-1}\alpha_i\sum_{j=i+1}^{n}\log\left(1+\frac{\sum_{\ell=1}^{i}\left\|\sum_{k=1}^{\ell}g_{k,j}\lambda_{k\ell}\sqrt{P'_k}\right\|^2}{\sigma_j^2}\right)$$
$$=\max_{\left\{\sum_{\ell=k}^{n-1}P_{k\ell}\le P_k\right\}}\ \max_{\{\phi_{k\ell}\in[0,2\pi)\}}\sum_{i=1}^{n-1}\alpha_i\sum_{j=i+1}^{n}\log\left(1+\frac{\sum_{\ell=1}^{i}\left\|\sum_{k=1}^{\ell}g_{k,j}\sqrt{P_{k\ell}}\ e^{\mathbf{i}\phi_{k\ell}}\right\|^2}{\sigma_j^2}\right)$$

where the last equation follows by letting $P_{k\ell}=\|\lambda_{k\ell}\|^2P'_k$ and $\phi_{k\ell}$ be the phase of $\lambda_{k\ell}$. □

IV. One Dimensional Networks

Consider a Gaussian network of n nodes $\mathcal{N}=\{1,2,\dots,n\}$ on a line, with coordinates $a_1<a_2<\cdots<a_n$. Let ρ_{ij} be the distance between Node i and Node j. Then $\rho_{ij}=|a_i-a_j|$. With the constraint of minimum separation distance $\rho_{\min}$, we have $a_{i+1}-a_i\ge\rho_{\min}$ for any $i=1,\dots,n-1$.

Let the gains be $g_{ij}:=\dfrac{e^{\mathbf{i}\theta_{ij}}}{\rho_{ij}^{\delta}}$, $i\ne j$ for some $\delta>0$ and $\theta_{ij}\in[0,2\pi)$, in the model (1). (The results for one-dimensional networks as well as others considered in subsequent sections extend trivially to the case where $g_{ij}:=\dfrac{ce^{\mathbf{i}\theta_{ij}}}{\rho_{ij}^{\delta}}$ for some pre-constant c). Let $\mathbb{E}\|Z_i(1)\|^2\equiv\sigma^2$ for $i\in\mathcal{N}$.

Define subsets $S_i^-:=\{1,\dots,i\}$ and $S_i^+:=\{i+1,\dots,n\}$, for $i=1,2,\dots,n-1$. Then it is easy to see that the total achieved transport is

$$\sum_{i=1}^{n-1}(a_{i+1}-a_i)\left(R^{(S_i^-)}+R^{(S_i^+)}\right).$$

Hence, applying Theorem 3.1 twice on $\sum_{i=1}^{n-1}(a_{i+1}-a_i)R^{(S_i^-)}$ and on $\sum_{i=1}^{n-1}(a_{i+1}-a_i)R^{(S_i^+)}$, we have the following bound on the transport capacity (using the inequality $\log(1+x)\le x\log e$ for $x>0$).

Theorem 4.1: The transport capacity is upper bounded by

$$C_{\mathrm{T}}\le\max_{\{P_{k\ell},\,\phi_{k\ell}\}}\frac{\log e}{\sigma^2}\sum_{i=1}^{n-1}(a_{i+1}-a_i)\times\left\{\sum_{j=i+1}^{n}\sum_{\ell=2}^{i+1}\left\|\sum_{k=1}^{\ell-1}\frac{e^{\mathbf{i}(\theta_{kj}+\phi_{k\ell})}\sqrt{P_{k\ell}}}{(a_j-a_k)^{\delta}}\right\|^2+\sum_{j=1}^{i}\sum_{\ell=i}^{n-1}\left\|\sum_{k=\ell+1}^{n}\frac{e^{\mathbf{i}(\theta_{kj}+\phi_{k\ell})}\sqrt{P_{k\ell}}}{(a_k-a_j)^{\delta}}\right\|^2\right\} \tag{28}$$

where $\phi_{k\ell}\in[0,2\pi)$, and $P_{k\ell}\ge 0$ are real numbers ($1\le k,\ell\le n$, and $k\ne\ell$) satisfying the total power constraint:

$$\sum_{k=1}^{n}\max\left\{\sum_{\ell=k+1}^{n}P_{k\ell}\,,\ \sum_{\ell=1}^{k-1}P_{k\ell}\right\}\le P_{\text{total}} \tag{29}$$

or the individual power constraint:

$$\max\left\{\sum_{\ell=k+1}^{n}P_{k\ell}\,,\ \sum_{\ell=1}^{k-1}P_{k\ell}\right\}\le P_{\text{ind}},\quad \forall k=1,\dots,n. \tag{30}$$

A. Expected Upper Bound with Random Phases

In this section, we develop upper bounds on the transport capacity under the assumption that the phases $\{\theta_{ij}\}$ are random variables, but are known to all the nodes, so that $\{\phi_{ij}\}$ can be designed based on $\{\theta_{ij}\}$. We assume that these random variables $\{\theta_{ij}\}$ are all uniformly distributed on $[0,2\pi)$ and also independent of each other.

First, by exchanging the order of summation, the bound (28) can be rewritten as

$$C_T\le\max_{\{P_{k\ell},\,\phi_{k\ell}\}}\frac{\log e}{\sigma^2}\left\{\sum_{\ell=2}^{n}\sum_{i=\ell-1}^{n-1}(a_{i+1}-a_i)\sum_{j=i+1}^{n}\left\|\sum_{k=1}^{\ell-1}\frac{e^{\mathbf{i}(\theta_{kj}+\phi_{k\ell})}\sqrt{P_{k\ell}}}{(a_j-a_k)^{\delta}}\right\|^2+\sum_{\ell=1}^{n-1}\sum_{i=1}^{\ell}(a_{i+1}-a_i)\sum_{j=1}^{i}\left\|\sum_{k=\ell+1}^{n}\frac{e^{\mathbf{i}(\theta_{kj}+\phi_{k\ell})}\sqrt{P_{k\ell}}}{(a_k-a_j)^{\delta}}\right\|^2\right\}=$$
$$\max_{\{P_{k\ell},\,\phi_{k\ell}\}}\frac{\log e}{\sigma^2}\left\{\sum_{\ell=2}^{n}\sum_{j=\ell}^{n}(a_j-a_{\ell-1})\left\|\sum_{k=1}^{\ell-1}\frac{e^{\mathbf{i}(\theta_{kj}+\phi_{k\ell})}\sqrt{P_{k\ell}}}{(a_j-a_k)^{\delta}}\right\|^2+\sum_{\ell=1}^{n-1}\sum_{j=1}^{\ell}(a_{\ell+1}-a_j)\left\|\sum_{k=\ell+1}^{n}\frac{e^{\mathbf{i}(\theta_{kj}+\phi_{k\ell})}\sqrt{P_{k\ell}}}{(a_k-a_j)^{\delta}}\right\|^2\right\}. \tag{31}$$

Using the inter-independence of $\{\theta_{ij}\}$, we have the following bound: for any $1 \le k, p \le \ell - 1$,

$$\begin{aligned}
&\mathbb{E}\sum_{j=\ell}^{n}(a_j - a_{\ell-1})\frac{e^{\mathbf{i}(\theta_{kj}+\phi_{k\ell})}\sqrt{P_{k\ell}}}{(a_j-a_k)^\delta}\frac{e^{-\mathbf{i}(\theta_{pj}+\phi_{p\ell})}\sqrt{P_{p\ell}}}{(a_j-a_p)^\delta}\\
&\le \mathbb{E}\left\|\sum_{j=\ell}^{n}(a_j - a_{\ell-1})\frac{e^{\mathbf{i}\theta_{kj}}}{(a_j-a_k)^\delta}\frac{e^{-\mathbf{i}\theta_{pj}}}{(a_j-a_p)^\delta}\sqrt{P_{k\ell}}\sqrt{P_{p\ell}}\right\|\\
&\le \left[\mathbb{E}\left\|\sum_{j=\ell}^{n}\frac{(a_j - a_{\ell-1})e^{\mathbf{i}\theta_{kj}}e^{-\mathbf{i}\theta_{pj}}}{(a_j-a_k)^\delta(a_j-a_p)^\delta}\sqrt{P_{k\ell}}\sqrt{P_{p\ell}}\right\|^2\right]^{\frac{1}{2}}\\
&= \left[\sum_{j=\ell}^{n}\frac{(a_j - a_{\ell-1})^2}{(a_j-a_k)^{2\delta}(a_j-a_p)^{2\delta}}P_{k\ell}P_{p\ell}\right]^{\frac{1}{2}}\\
&\le \left[\sum_{j=\ell}^{n}\frac{(a_j - a_{\ell-1})^2}{(a_j-a_k)^{2\delta}(a_j-a_p)^{2\delta}}\right]^{\frac{1}{2}}\frac{P_{k\ell}+P_{p\ell}}{2}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\mathbb{E}\sum_{j=\ell}^{n}(a_j - a_{\ell-1})\left\|\sum_{k=1}^{\ell-1}\frac{e^{\mathbf{i}(\theta_{kj}+\phi_{k\ell})}\sqrt{P_{k\ell}}}{(a_j-a_k)^\delta}\right\|^2\\
&= \sum_{k=1}^{\ell-1}\sum_{p=1}^{\ell-1}\mathbb{E}\sum_{j=\ell}^{n}(a_j - a_{\ell-1})\frac{e^{\mathbf{i}(\theta_{kj}+\phi_{k\ell})}\sqrt{P_{k\ell}}}{(a_j-a_k)^\delta}\\
&\qquad\qquad\times\frac{e^{-\mathbf{i}(\theta_{pj}+\phi_{p\ell})}\sqrt{P_{p\ell}}}{(a_j-a_p)^\delta}\\
&\le \sum_{k=1}^{\ell-1}\sum_{p=1}^{\ell-1}\left[\sum_{j=\ell}^{n}\frac{(a_j - a_{\ell-1})^2}{(a_j-a_k)^{2\delta}(a_j-a_p)^{2\delta}}\right]^{\frac{1}{2}}\frac{P_{k\ell}+P_{p\ell}}{2}\\
&= \sum_{k=1}^{\ell-1}\sum_{p=1}^{\ell-1}\left[\sum_{j=\ell}^{n}\frac{(a_j - a_{\ell-1})^2}{(a_j-a_k)^{2\delta}(a_j-a_p)^{2\delta}}\right]^{\frac{1}{2}}P_{k\ell}. \qquad (32)
\end{aligned}$$

And similarly,

$$\begin{aligned}
&\mathbb{E}\sum_{j=1}^{\ell}(a_{\ell+1} - a_j)\left\|\sum_{k=\ell+1}^{n}\frac{e^{\mathbf{i}(\theta_{kj}+\phi_{k\ell})}\sqrt{P_{k\ell}}}{(a_k-a_j)^\delta}\right\|^2\\
&\le \sum_{k=\ell+1}^{n}\sum_{p=\ell+1}^{n}\left[\sum_{j=1}^{\ell}\frac{(a_{\ell+1} - a_j)^2}{(a_k-a_j)^{2\delta}(a_p-a_j)^{2\delta}}\right]^{\frac{1}{2}}P_{k\ell}.
\end{aligned}$$

Therefore, by (31), we have

$$\begin{aligned}
&\mathbb{E}C_{\mathrm{T}} \le \max_{\{P_{k\ell}\}}\frac{\log e}{\sigma^2}\times \qquad (33)\\
&\left\{\sum_{\ell=2}^{n}\sum_{k=1}^{\ell-1}\sum_{p=1}^{\ell-1}\left[\sum_{j=\ell}^{n}\frac{(a_j - a_{\ell-1})^2}{(a_j-a_k)^{2\delta}(a_j-a_p)^{2\delta}}\right]^{\frac{1}{2}}P_{k\ell}\right.\\
&\left.+\sum_{\ell=1}^{n-1}\sum_{k=\ell+1}^{n}\sum_{p=\ell+1}^{n}\left[\sum_{j=1}^{\ell}\frac{(a_{\ell+1} - a_j)^2}{(a_k-a_j)^{2\delta}(a_p-a_j)^{2\delta}}\right]^{\frac{1}{2}}P_{k\ell}\right\}.
\end{aligned}$$

Now, under the total power constraint (29), we can show that $\mathbb{E}C_{\mathrm{T}}$ is upper bounded by the total power P_{total} to a constant factor if the coefficients of all the $P_{k\ell}$ in (33) are uniformly bounded by a constant. That is, we need to show that the following terms are uniformly bounded:

$$\sum_{p=1}^{\ell-1}\left[\sum_{j=\ell}^{n}\frac{(a_j - a_{\ell-1})^2}{(a_j-a_k)^{2\delta}(a_j-a_p)^{2\delta}}\right]^{\frac{1}{2}}, \quad \forall 1 \le k < \ell \le n, \qquad (34)$$

$$\sum_{p=\ell+1}^{n}\left[\sum_{j=1}^{\ell}\frac{(a_{\ell+1} - a_j)^2}{(a_k-a_j)^{2\delta}(a_p-a_j)^{2\delta}}\right]^{\frac{1}{2}}, \quad \forall 1 \le \ell < k \le n. \qquad (35)$$

To this end, first, for $\delta > 5/4$, we have the following uniform upper bound for all the terms in (34):

$$\begin{aligned}
&\max_{1\le k<\ell\le n}\sum_{p=1}^{\ell-1}\left[\sum_{j=\ell}^{n}(a_j - a_{\ell-1})^2\frac{1}{(a_j-a_k)^{2\delta}}\frac{1}{(a_j-a_p)^{2\delta}}\right]^{\frac{1}{2}}\\
&= \max_{1<\ell\le n}\sum_{p=1}^{\ell-1}\left[\sum_{j=\ell}^{n}\frac{1}{(a_j-a_{\ell-1})^{2\delta-2}}\frac{1}{(a_j-a_p)^{2\delta}}\right]^{\frac{1}{2}}\\
&\le \max_{1<\ell\le n}\sum_{p=1}^{\ell-1}\left[\sum_{j=\ell}^{n}\frac{1}{(a_j-a_{\ell-1})^{2\delta-2}}\right.\\
&\qquad\qquad\left.\times\frac{1}{(a_j-a_{\ell-1})^{\frac{1}{2}}(a_\ell-a_p)^{2\delta-\frac{1}{2}}}\right]^{\frac{1}{2}}\\
&\le \max_{1<\ell\le n}\sum_{p=1}^{\ell-1}\frac{1}{(a_\ell-a_p)^{\delta-\frac{1}{4}}}\left[\sum_{j=\ell}^{n}\frac{1}{(a_j-a_{\ell-1})^{2\delta-\frac{3}{2}}}\right]^{\frac{1}{2}}\\
&\overset{(a)}{\le} \frac{1}{\rho_{\min}^{2\delta-1}}\max_{1<\ell\le n}\sum_{p=1}^{\ell-1}\frac{1}{(\ell-p)^{\delta-\frac{1}{4}}}\left[\sum_{j=\ell}^{n}\frac{1}{(j-\ell+1)^{2\delta-\frac{3}{2}}}\right]^{\frac{1}{2}}\\
&\overset{(b)}{\le} \frac{1}{\rho_{\min}^{2\delta-1}}\frac{\delta-\frac{1}{4}}{\delta-\frac{5}{4}}\left(\frac{2\delta-\frac{3}{2}}{2\delta-\frac{5}{2}}\right)^{\frac{1}{2}} = \frac{1}{\rho_{\min}^{2\delta-1}}\frac{(4\delta-1)(4\delta-3)^{\frac{1}{2}}}{(4\delta-5)^{\frac{3}{2}}}
\end{aligned}$$

where the inequality (a) follows from the minimum distance constraint, and the inequality (b) follow from the fact that for any real $a > 0$ and $\beta > 1$,

$$\sum_{l=0}^{\infty}\frac{1}{(a+l)^\beta} \le \frac{1}{a^\beta}+\int_0^\infty\frac{1}{(a+x)^\beta}dx \le \frac{1}{a^\beta}+\frac{1}{(\beta-1)a^{\beta-1}}. \qquad (36)$$

Similarly, we can prove that the same upper bound holds for all the terms in (35). This leads to the following theorem.

Theorem 4.2: Let the phases $\{\theta_{ij}\}$ be independent of each other and uniformly distributed on $[0, 2\pi)$, but their realizations are known to all the nodes. Under the total power constraint, for $\delta > 5/4$, the expected transport capacity is always upper bounded by the total power to within a constant factor:

$$\mathbb{E}\,C_{\mathrm{T}} \le \frac{\bar{c}_1(\delta)\log e}{\sigma^2\rho_{\min}^{2\delta-1}}P_{\text{total}} \qquad (37)$$

where

$$\bar{c}_1(\delta) = \frac{2(4\delta-1)(4\delta-3)^{\frac{1}{2}}}{(4\delta-5)^{\frac{3}{2}}}. \qquad (38)$$

The following theorem establishing a linear scaling law under the individual power constraint follows immediately by noting that $P_{\text{total}} = n \cdot P_{\text{ind}}$.

Theorem 4.3: Let the phases $\{\theta_{ij}\}$ be independent of each other and uniformly distributed on $[0, 2\pi)$, but their realizations are known to all the nodes. Under the individual power constraint, for $\delta > 5/4$, the expected transport capacity is always upper bounded by the number of nodes n to a constant factor:

$$\mathbb{E}\, C_{\text{T}} \le \frac{\bar{c}_1(\delta)\log e}{\sigma^2 \rho_{\min}^{2\delta-1}} P_{\text{ind}} \cdot n \tag{39}$$

where $\bar{c}_1$ is defined in (38).

B. Uniform Upper Bounds Irrespective of Phases

In this section, we consider uniform upper bounds on the transport capacity for all possible realizations of the phases $\{\theta_{ij}\}$.

Our first result, the following theorem, follows immediately from the bound (28).

Theorem 4.4: The transport capacity is upper bounded by

$$C_{\text{T}} \le \max_{\{P_{k\ell}\}} \frac{\log e}{\sigma^2} \sum_{i=1}^{n-1} (a_{i+1} - a_i) \left\{ \sum_{j=i+1}^{n} \sum_{\ell=2}^{i+1} \left[\sum_{k=1}^{\ell-1} \frac{\sqrt{P_{k\ell}}}{(a_j - a_k)^\delta} \right]^2 + \sum_{j=1}^{i} \sum_{\ell=i}^{n-1} \left[\sum_{k=\ell+1}^{n} \frac{\sqrt{P_{k\ell}}}{(a_k - a_j)^\delta} \right]^2 \right\} \tag{40}$$

where the nonnegative powers $P_{k\ell}$ ($1 \le k, \ell \le n$, and $k \ne \ell$) satisfy the total power constraint:

$$\sum_{k=1}^{n} \max\left\{ \sum_{\ell=k+1}^{n} P_{k\ell}\,, \sum_{\ell=1}^{k-1} P_{k\ell} \right\} \le P_{\text{total}} \tag{41}$$

or the individual power constraint:

$$\max\left\{ \sum_{\ell=k+1}^{n} P_{k\ell}\,, \sum_{\ell=1}^{k-1} P_{k\ell} \right\} \le P_{\text{ind}} \quad \forall k = 1, \ldots, n. \tag{42}$$

The bound (40) can be weakened by using the the following inequalities:

$$\begin{aligned}
\left[\sum_{k=1}^{\ell-1} \frac{\sqrt{P_{k\ell}}}{(a_j - a_k)^\delta} \right]^2 &= \sum_{k=1}^{\ell-1} \frac{\sqrt{P_{k\ell}}}{(a_j - a_k)^\delta} \sum_{p=1}^{\ell-1} \frac{\sqrt{P_{p\ell}}}{(a_j - a_p)^\delta} \\
&\le \sum_{k=1}^{\ell-1} \frac{1}{(a_j - a_k)^\delta} \sum_{p=1}^{\ell-1} \frac{1}{(a_j - a_p)^\delta} \frac{P_{k\ell} + P_{p\ell}}{2} \\
&= \sum_{k=1}^{\ell-1} \frac{1}{(a_j - a_k)^\delta} \sum_{p=1}^{\ell-1} \frac{1}{(a_j - a_p)^\delta} P_{k\ell}
\end{aligned} \tag{43}$$

and a similarly obtained one,

$$\left[\sum_{k=\ell+1}^{n} \frac{\sqrt{P_{k\ell}}}{(a_k - a_j)^\delta} \right]^2 \le \sum_{k=\ell+1}^{n} \frac{1}{(a_k - a_j)^\delta} \sum_{p=\ell+1}^{n} \frac{1}{(a_p - a_j)^\delta} P_{k\ell}.$$

Now the upper bound (40) can be weakened to

$$\begin{aligned}
C_{\text{T}} &\le \max_{\{P_{k\ell}\}} \frac{\log e}{2\sigma^2} \sum_{i=1}^{n-1} (a_{i+1} - a_i) \\
&\quad \times \left\{ \sum_{j=i+1}^{n} \sum_{\ell=2}^{i+1} \sum_{k=1}^{\ell-1} \frac{1}{(a_j - a_k)^\delta} \sum_{p=1}^{\ell-1} \frac{1}{(a_j - a_p)^\delta} P_{k\ell} \right. \\
&\qquad \left. + \sum_{j=1}^{i} \sum_{\ell=i}^{n-1} \sum_{k=\ell+1}^{n} \frac{1}{(a_k - a_j)^\delta} \sum_{p=\ell+1}^{n} \frac{1}{(a_p - a_j)^\delta} P_{k\ell} \right\} \\
&= \max_{\{P_{k\ell}\}} \frac{\log e}{2\sigma^2} \left\{ \sum_{\ell=2}^{n} \sum_{k=1}^{\ell-1} \sum_{i=\ell-1}^{n-1} (a_{i+1} - a_i) \sum_{j=i+1}^{n} \frac{1}{(a_j - a_k)^\delta} \right. \\
&\quad \times \sum_{p=1}^{\ell-1} \frac{1}{(a_j - a_p)^\delta} P_{k\ell} + \sum_{\ell=1}^{n-1} \sum_{k=\ell+1}^{n} \sum_{i=1}^{\ell} (a_{i+1} - a_i) \\
&\qquad \left. \times \sum_{j=1}^{i} \frac{1}{(a_k - a_j)^\delta} \sum_{p=\ell+1}^{n} \frac{1}{(a_p - a_j)^\delta} P_{k\ell} \right\} \\
&= \max_{\{P_{k\ell}\}} \frac{\log e}{2\sigma^2} \left\{ \sum_{\ell=2}^{n} \sum_{k=1}^{\ell-1} \sum_{j=\ell}^{n} \frac{(a_j - a_{\ell-1})}{(a_j - a_k)^\delta} \sum_{p=1}^{\ell-1} \frac{1}{(a_j - a_p)^\delta} P_{k\ell} \right. \\
&\qquad \left. + \sum_{\ell=1}^{n-1} \sum_{k=\ell+1}^{n} \sum_{j=1}^{\ell} \frac{(a_{\ell+1} - a_j)}{(a_k - a_j)^\delta} \sum_{p=\ell+1}^{n} \frac{1}{(a_p - a_j)^\delta} P_{k\ell} \right\}
\end{aligned} \tag{44}$$

where the last two equalities follow by exchanging the order of summation.

Now, under the total power constraint (41), we can show that C_{T} is upper bounded by the total power P_{total} to within a constant factor if the coefficients of all the $P_{k\ell}$ in (44) are uniformly bounded by a constant. Hence we proceed to show that the following terms are uniformly bounded:

$$\sum_{j=\ell}^{n} \frac{(a_j - a_{\ell-1})}{(a_j - a_k)^\delta} \sum_{p=1}^{\ell-1} \frac{1}{(a_j - a_p)^\delta}, \ \forall 1 \le k < \ell \le n; \tag{45}$$

$$\sum_{j=1}^{\ell} \frac{(a_{\ell+1} - a_j)}{(a_k - a_j)^\delta} \sum_{p=\ell+1}^{n} \frac{1}{(a_p - a_j)^\delta}, \ \forall 1 \le \ell < k \le n. \tag{46}$$

First, for $\delta > 3/2$, we have the following uniform upper bound for all the terms in (45):

$$\begin{aligned}
&\max_{1 \le k < \ell \le n} \sum_{j=\ell}^{n} \frac{(a_j - a_{\ell-1})}{(a_j - a_k)^\delta} \sum_{p=1}^{\ell-1} \frac{1}{(a_j - a_p)^\delta} \\
&= \max_{1 < \ell \le n} \sum_{j=\ell}^{n} \frac{(a_j - a_{\ell-1})}{(a_j - a_{\ell-1})^\delta} \sum_{p=1}^{\ell-1} \frac{1}{(a_j - a_p)^\delta} \\
&= \max_{1 < \ell \le n} \sum_{j=\ell}^{n} \frac{1}{(a_j - a_{\ell-1})^{\delta-1}} \sum_{p=1}^{\ell-1} \frac{1}{(a_j - a_p)^\delta} \\
&\overset{(a)}{\le} \frac{1}{\rho_{\min}^{2\delta-1}} \max_{1 < \ell \le n} \sum_{j=\ell}^{n} \frac{1}{[j - (\ell-1)]^{\delta-1}} \sum_{p=1}^{\ell-1} \frac{1}{(j-p)^\delta}
\end{aligned}$$

$$\leq \frac{1}{\rho_{\min}^{2\delta-1}} \max_{1<\ell\leq n} \sum_{j=\ell}^{n} \frac{1}{[j-(\ell-1)]^{\delta-1}} \times \left\{ \frac{1}{[j-(\ell-1)]^{\delta}} + \frac{1}{(\delta-1)[j-(\ell-1)]^{\delta-1}} \right\}$$
$$\leq \frac{1}{\rho_{\min}^{2\delta-1}} \left[1 + \frac{3}{2\delta-2} + \frac{1}{(\delta-1)(2\delta-3)} \right].$$

where the inequality (a) follows from the minimum distance constraint, and the last two inequalities follow from the fact (36).

Similarly, we can prove that the same upper bound holds for all the terms in (46). This leads to the following theorem.

Theorem 4.5: Under the total power constraint, for $\delta > 3/2$, the transport capacity is always upper bounded by the total power to within a constant factor:

$$C_{\text{T}} \leq \frac{c_1(\delta) \log e}{\sigma^2 \rho_{\min}^{2\delta-1}} P_{\text{total}} \tag{47}$$

where

$$c_1(\delta) = 2\left[1 + \frac{3}{2\delta-2} + \frac{1}{(\delta-1)(2\delta-3)} \right]. \tag{48}$$

The following theorem establishing a linear scaling law under the individual power constraint follows immediately by setting $P_{\text{total}} = n \cdot P_{\text{ind}}$.

Theorem 4.6: Under the individual power constraint, for $\delta > 3/2$, the transport capacity is always upper bounded by the number of nodes n to within a constant factor:

$$C_{\text{T}} \leq \frac{c_1(\delta) \log e}{\sigma^2 \rho_{\min}^{2\delta-1}} P_{\text{ind}} \cdot n \tag{49}$$

where c_1 is defined in (48).

Next, we show that $\delta > 3/2$ is almost the weakest requirement for any linear scaling law that we can prove with Theorem 4.1. That is, we will show that for any $\delta < 3/2$, under the individual power constraint (30), there exists a topology $a_1 < a_2 < \cdots < a_n$ of the network, and $\{\theta_{ij}\}$, such that the right-hand-side (R.H.S.) of (28) is not upper bounded by the number of nodes n to within any constant factor, i.e.,

$$\frac{\text{R.H.S. of (28)}}{n} \to \infty, \quad \text{as } n \to \infty. \tag{50}$$

Consider any $\rho_0 \geq \max\{\rho_{\min}, 1\}$. Let $a_{i+1} - a_i \equiv \rho_0$, $i = 1, 2, \ldots, n-1$, and let $\theta_{ij} \equiv 0$, $\forall i \neq j$. Choose

$$P_{k\ell} \equiv \frac{P_{\text{ind}}}{n-1} \quad \text{and} \quad \phi_{k\ell} \equiv 0 \quad \text{for all } k \neq \ell, \tag{51}$$

where it is easy to check that the individual power constraint (30) is satisfied. For this choice, the R.H.S. of (28) simplifies to

$$\frac{P_{\text{ind}} \log e}{(n-1)\sigma^2 \rho_0^{2\delta-1}} \sum_{i=1}^{n-1} \left\{ \sum_{j=i+1}^{n} \sum_{\ell=2}^{i+1} \left[\sum_{k=1}^{\ell-1} \frac{1}{(j-k)^{\delta}} \right]^2 + \sum_{j=1}^{i} \sum_{\ell=i}^{n-1} \left[\sum_{k=\ell+1}^{n} \frac{1}{(k-j)^{\delta}} \right]^2 \right\}.$$

Let

$$B(n,\delta) := \frac{1}{(n-1)\rho_0^{2\delta-1}} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sum_{\ell=2}^{i+1} \left[\sum_{k=1}^{\ell-1} \frac{1}{(j-k)^{\delta}} \right]^2.$$

Since $\rho_0 \geq 1$, $B(n,\delta)$ is a decreasing function of δ. Therefore, we only need to show that for any $1 < \delta < 3/2$,

$$\frac{B(n,\delta)}{n} \to \infty, \quad \text{as } n \to \infty. \tag{52}$$

Although a little messy, the following straightforward calculations lead directly to (52). First,

$$B(n,\delta) \geq \frac{1}{(n-1)\rho_0^{2\delta-1}} \sum_{i=\lceil n/2 \rceil}^{n-1} \sum_{j=i+1}^{n} \sum_{\ell=\lceil j/2 \rceil+1}^{i+1} \left[\sum_{k=1}^{\ell-1} \frac{1}{(j-k)^{\delta}} \right]^2 \geq \frac{1}{(n-1)\rho_0^{2\delta-1}} \sum_{i=\lceil n/2 \rceil}^{n-1} \sum_{j=i+1}^{n} \sum_{\ell=\lceil j/2 \rceil+1}^{i+1} \left[\int_0^{\ell-1} \frac{1}{(j-x)^{\delta}} dx \right]^2.$$

Since $j-(\ell-1) < j/2$ for any $\lceil j/2 \rceil + 1 \leq \ell \leq i+1$, we have

$$\sum_{\ell=\lceil j/2 \rceil+1}^{i+1} \left[\int_0^{\ell-1} \frac{1}{(j-x)^{\delta}} dx \right]^2$$
$$= \sum_{\ell=\lceil j/2 \rceil+1}^{i+1} \left\{ \frac{1}{(\delta-1)[j-(\ell-1)]^{\delta-1}} - \frac{1}{(\delta-1)[2(j/2)]^{\delta-1}} \right\}^2$$
$$\geq \sum_{\ell=\lceil j/2 \rceil+1}^{i+1} \left\{ \frac{1}{(\delta-1)} \left(1 - \frac{1}{2^{\delta-1}} \right) \frac{1}{[j-(\ell-1)]^{\delta-1}} \right\}^2$$
$$= \frac{1}{(\delta-1)^2} \left(1 - \frac{1}{2^{\delta-1}} \right)^2 \sum_{\ell=\lceil j/2 \rceil+1}^{i+1} \frac{1}{[j-(\ell-1)]^{2\delta-2}}$$
$$\geq \frac{1}{(\delta-1)^2} \left(1 - \frac{1}{2^{\delta-1}} \right)^2 \int_{x=\lceil j/2 \rceil}^{i} \frac{1}{[j-x]^{2\delta-2}} dx \tag{53}$$
$$= \alpha_1(\delta) \left[(j - \lceil j/2 \rceil)^{3-2\delta} - (j-i)^{3-2\delta} \right]$$

where

$$\alpha_1(\delta) = \frac{1}{(\delta-1)^2} \left(1 - \frac{1}{2^{\delta-1}} \right)^2 \frac{1}{(3-2\delta)} > 0.$$

Therefore

$$B(n,\delta) \geq \frac{1}{(n-1)\rho_0^{2\delta-1}} \alpha_1(\delta) \sum_{i=\lceil n/2 \rceil}^{n-1} \sum_{j=i+1}^{n} \left[(j - \lceil j/2 \rceil)^{3-2\delta} - (j-i)^{3-2\delta} \right]$$
$$\geq \frac{1}{(n-1)\rho_0^{2\delta-1}} \alpha_1(\delta) \sum_{i=\lceil n/2 \rceil}^{n-1} \sum_{j=i+1}^{n} \left[(j/2-1)^{3-2\delta} - (j-i)^{3-2\delta} \right]. \tag{54}$$

Again, approximation of a summation by an integral in (54) leads to

$$\sum_{i=\lceil n/2\rceil}^{n-1}\sum_{j=i+1}^{n}\left[(j/2-1)^{3-2\delta}-(j-i)^{3-2\delta}\right]$$
$$\geq \sum_{i=\lceil n/2\rceil}^{n-1}\left[\int_{i+1}^{n}(x/2-1)^{3-2\delta}dx-\int_{i+1}^{n+1}(x-i)^{3-2\delta}dx\right]$$
$$=\frac{1}{4-2\delta}\sum_{i=\lceil n/2\rceil}^{n-1}\Big\{2(n/2-1)^{4-2\delta}-2[(i+1)/2-1]^{4-2\delta}$$
$$-(n+1-i)^{4-2\delta}+1\Big\}$$
$$\geq \frac{1}{4-2\delta}\left\{2\left(\frac{n}{2}-1\right)^{5-2\delta}-\int_{\lceil n/2\rceil}^{n}2\left[\frac{(x+1)}{2}-1\right]^{4-2\delta}dx\right.$$
$$\left.-\int_{\lceil n/2\rceil-1}^{n-1}(n+1-x)^{4-2\delta}dx+1\right\}$$
$$\geq \frac{1}{4-2\delta}\left\{\frac{1}{2^{4-2\delta}}(n-2)^{5-2\delta}-\frac{1}{5-2\delta}\frac{1}{2^{4-2\delta}}(n-1)^{5-2\delta}\right.$$
$$\left.-\frac{1}{5-2\delta}(n-\lceil n/2\rceil+2)^{5-2\delta}+1\right\}$$
$$\geq \frac{1}{4-2\delta}\left\{\frac{1}{2^{4-2\delta}}(n-2)^{5-2\delta}-\frac{1}{5-2\delta}\frac{1}{2^{4-2\delta}}(n-1)^{5-2\delta}\right.$$
$$\left.-\frac{1}{5-2\delta}\frac{1}{2^{5-2\delta}}(n+4)^{5-2\delta}+1\right\}$$
$$=\alpha_2(\delta)n^{5-2\delta}+o(1) \tag{55}$$

where

$$\alpha_2(\delta)=\frac{1}{4-2\delta}\left\{\frac{1}{2^{4-2\delta}}-\frac{1}{5-2\delta}\frac{1}{2^{4-2\delta}}-\frac{1}{5-2\delta}\frac{1}{2^{5-2\delta}}\right\}$$
$$>0.$$

Finally, by (54) and (55), we obtain that for any $1<\delta<3/2$,

$$\frac{B(n,\delta)}{n}\geq \frac{1}{n(n-1)\rho_0^{2\delta-1}}\,\alpha_1(\delta)\alpha_2(\delta)n^{5-2\delta}+o(1)$$
$$\to\infty,\quad \text{as } n\to\infty.$$

The above example shows that only with Theorem 4.1, under the individual power constraint, we cannot expect to prove that the transport capacity is always upper bounded by the number of nodes n to within a constant factor for any $\delta<3/2$.

As a fringe benefit, it also shows that with only Theorem 4.1, under the total power constraint, we cannot expect to prove that the transport capacity is always upper bounded by the total power to within a constant factor, for any $\delta<3/2$.

V. Two Dimensional Networks

Consider a Gaussian network of n nodes $\mathcal{N}=\{1,2,\ldots,n\}$ on a plane, with coordinates (a_i,b_i), $i=1,\ldots,n$. Let ρ_{ij} be the distance between Node i and Node j. Then

$$\rho_{ij}=\sqrt{(a_i-a_j)^2+(b_i-b_j)^2}.$$

With the constraint of minimum separation distance $\rho_{\min}$, we have $\rho_{ij}\geq\rho_{\min}$ for any $i\neq j$.

Let the gains $g_{ij}=\dfrac{e^{\mathbf{i}\theta_{ij}}}{\rho_{ij}^{\delta}}$, $i\neq j$ for some $\delta>0$ and $\theta_{ij}\in[0,2\pi)$ in the model (1). Set $\mathbb{E}\|Z_i(t)\|^2\equiv\sigma^2$ for $i\in\mathcal{N}$.

We order the nodes horizontally and vertically as follows. Let $(u_1,u_2,\ldots,u_n)$ and $(v_1,v_2,\ldots,v_n)$ be two permutations of $(1,2,\ldots,n)$ such that

$$a_{u_1}\leq a_{u_2}\leq\cdots\leq a_{u_n},\quad\text{and}\quad b_{v_1}\leq b_{v_2}\leq\cdots\leq b_{v_n}.$$

For $i=1,2,\ldots,n-1$, define subsets

$$\mathcal{A}_i^-:=\{u_1,\ldots,u_i\},\quad\text{and}\quad \mathcal{A}_i^+:=\{u_{i+1},\ldots,u_n\}$$
$$\mathcal{B}_i^-:=\{v_1,\ldots,v_i\},\quad\text{and}\quad \mathcal{B}_i^+:=\{v_{i+1},\ldots,v_n\}$$

Then it is easy to see geometrically that the total achieved transport is upper bounded[1] by

$$\bar{Q}=\sum_{i=1}^{n-1}(a_{u_{i+1}}-a_{u_i})\left(R^{(\mathcal{A}_i^-)}+R^{(\mathcal{A}_i^+)}\right)$$
$$+\sum_{i=1}^{n-1}(b_{v_{i+1}}-b_{v_i})\left(R^{(\mathcal{B}_i^-)}+R^{(\mathcal{B}_i^+)}\right)$$

and also is lower bounded by $\dfrac{\sqrt{2}}{2}\bar{Q}$.

Hence, applying Theorem 3.1 four times on $\mathcal{A}_i^-:=\{u_1,\ldots,u_i\}$, on $\mathcal{A}_i^+:=\{u_{i+1},\ldots,u_n\}$, on $\mathcal{B}_i^-:=\{v_1,\ldots,v_i\}$, and on $\mathcal{B}_i^+:=\{v_{i+1},\ldots,v_n\}$, we have the following bound on the transport capacity. We use the inequality $\log(1+x)\leq x\log e$ for $x>0$.

Theorem 5.1: The transport capacity is upper bounded by

$$C_{\mathrm{T}}\leq\max_{\{P_{u_ku_\ell},\,\phi_{u_ku_\ell}\}}\frac{\log e}{\sigma^2}\sum_{i=1}^{n-1}(a_{u_{i+1}}-a_{u_i}) \tag{56}$$
$$\left\{\sum_{j=i+1}^{n}\sum_{\ell=2}^{i+1}\left\|\sum_{k=1}^{\ell-1}\frac{e^{\mathbf{i}(\theta_{u_ku_j}+\phi_{u_ku_\ell})}\sqrt{P_{u_ku_\ell}}}{\rho_{u_ku_j}^{\delta}}\right\|^2\right.$$
$$\left.+\sum_{j=1}^{i}\sum_{\ell=i}^{n-1}\left\|\sum_{k=\ell+1}^{n}\frac{e^{\mathbf{i}(\theta_{u_ku_j}+\phi_{u_ku_\ell})}\sqrt{P_{u_ku_\ell}}}{\rho_{u_ku_j}^{\delta}}\right\|^2\right\}$$
$$+\max_{\{P'_{v_kv_\ell},\,\phi'_{v_kv_\ell}\}}\frac{\log e}{\sigma^2}\sum_{i=1}^{n-1}(b_{v_{i+1}}-b_{v_i})$$
$$\left\{\sum_{j=i+1}^{n}\sum_{\ell=2}^{i+1}\left\|\sum_{k=1}^{\ell-1}\frac{e^{\mathbf{i}(\theta_{v_kv_j}+\phi'_{v_kv_\ell})}\sqrt{P'_{v_kv_\ell}}}{\rho_{v_kv_j}^{\delta}}\right\|^2\right.$$
$$\left.+\sum_{j=1}^{i}\sum_{\ell=i}^{n-1}\left\|\sum_{k=\ell+1}^{n}\frac{e^{\mathbf{i}(\theta_{v_kv_j}+\phi'_{v_kv_\ell})}\sqrt{P'_{v_kv_\ell}}}{\rho_{v_kv_j}^{\delta}}\right\|^2\right\}$$

where $\phi_{u_ku_\ell}\in[0,2\pi)$, $\phi'_{v_kv_\ell}\in[0,2\pi)$, and $P_{u_ku_\ell}\geq0$, $P'_{v_kv_\ell}\geq0$ are real numbers ($1\leq k,\ell\leq n$, and $k\neq\ell$) satisfying the total power constraint:

$$\begin{cases}\sum_{k=1}^{n}\max\left\{\sum_{\ell=k+1}^{n}P_{u_ku_\ell},\,\sum_{\ell=1}^{k-1}P_{u_ku_\ell}\right\}\leq P_{\text{total}}\\ \sum_{k=1}^{n}\max\left\{\sum_{\ell=k+1}^{n}P'_{v_kv_\ell},\,\sum_{\ell=1}^{k-1}P'_{v_kv_\ell}\right\}\leq P_{\text{total}}\end{cases} \tag{57}$$

[1] An exact expression for the achieved transport can be obtained by random cuts omni-directionally on the plane, as in [11] and [16]. But here, for the application of Theorem 3.1, the nodes need to be ordered.

or the individual power constraint: For all $k = 1, \ldots, n$,

$$\begin{cases} \max\left\{\sum_{\ell=k+1}^{n} P_{u_k u_\ell}\,,\ \sum_{\ell=1}^{k-1} P_{u_k u_\ell}\right\} \leq P_{\text{ind}} \\ \max\left\{\sum_{\ell=k+1}^{n} P'_{v_k v_\ell}\,,\ \sum_{\ell=1}^{k-1} P'_{v_k v_\ell}\right\} \leq P_{\text{ind}} \end{cases} \tag{58}$$

A. Expected Upper Bound with Random Phases

In this section, we determine upper bounds on the transport capacity under the assumption that the phases $\{\theta_{ij}\}$ are random variables, but are known to all the nodes, so that $\{\phi_{ij}\}$ can be designed based on $\{\theta_{ij}\}$. We assume that these random variables $\{\theta_{ij}\}$ are all uniformly distributed on $[0, 2\pi)$, and also independent of each other.

Similar to the one-dimensional case (31), by exchanging the order of summation, we can rewrite the first part of the R.H.S. of (56) as,

$$\sum_{i=1}^{n-1}(a_{u_{i+1}} - a_{u_i}) \sum_{j=i+1}^{n} \sum_{\ell=2}^{i+1} \left\| \sum_{k=1}^{\ell-1} \frac{e^{\mathbf{i}(\theta_{u_k u_j} + \phi_{u_k u_\ell})}\sqrt{P_{u_k u_\ell}}}{\rho^{\delta}_{u_k u_j}} \right\|^2$$
$$= \sum_{\ell=2}^{n} \sum_{j=\ell}^{n} (a_{u_j} - a_{u_{\ell-1}}) \left\| \sum_{k=1}^{\ell-1} \frac{e^{\mathbf{i}(\theta_{u_k u_j} + \phi_{u_k u_\ell})}\sqrt{P_{u_k u_\ell}}}{\rho^{\delta}_{u_k u_j}} \right\|^2.$$

Then similar to (32), using the inter-independence of $\{\theta_{ij}\}$, we have the following bound:

$$\mathbb{E} \sum_{j=\ell}^{n} (a_{u_j} - a_{u_{\ell-1}}) \left\| \sum_{k=1}^{\ell-1} \frac{e^{\mathbf{i}(\theta_{u_k u_j} + \phi_{u_k u_\ell})}\sqrt{P_{u_k u_\ell}}}{\rho^{\delta}_{u_k u_j}} \right\|^2$$
$$\leq \sum_{k=1}^{\ell-1} \sum_{p=1}^{\ell-1} \left[\sum_{j=\ell}^{n} \frac{(a_{u_j} - a_{u_{\ell-1}})^2}{\rho^{2\delta}_{u_k u_j} \rho^{2\delta}_{u_p u_j}} \right]^{\frac{1}{2}} P_{u_k u_\ell}.$$

Similarly, we can obtain bounds for the other three parts in the R.H.S. of (56). Therefore, as in the one-dimensional case (33), we have

$$\mathbb{E}C_{\text{T}} \leq \max_{\{P_{u_k u_\ell}\}} \frac{\log e}{\sigma^2} \left\{ \sum_{\ell=2}^{n} \sum_{k=1}^{\ell-1} \sum_{p=1}^{\ell-1} \left[\sum_{j=\ell}^{n} \frac{(a_{u_j} - a_{u_{\ell-1}})^2}{\rho^{2\delta}_{u_k u_j} \rho^{2\delta}_{u_p u_j}} \right]^{\frac{1}{2}} \right.$$
$$\left. P_{u_k u_\ell} + \sum_{\ell=1}^{n-1} \sum_{k=\ell+1}^{n} \sum_{p=\ell+1}^{n} \left[\sum_{j=1}^{\ell} \frac{(a_{u_{\ell+1}} - a_{u_j})^2}{\rho^{2\delta}_{u_k u_j} \rho^{2\delta}_{u_p u_j}} \right]^{\frac{1}{2}} P_{u_k u_\ell} \right\}$$
$$+ \max_{\{P'_{v_k v_\ell}\}} \frac{\log e}{\sigma^2} \left\{ \sum_{\ell=2}^{n} \sum_{k=1}^{\ell-1} \sum_{p=1}^{\ell-1} \left[\sum_{j=\ell}^{n} \frac{(a_{v_j} - a_{v_{\ell-1}})^2}{\rho^{2\delta}_{v_k v_j} \rho^{2\delta}_{v_p v_j}} \right]^{\frac{1}{2}} P'_{v_k v_\ell} \right.$$
$$\left. + \sum_{\ell=1}^{n-1} \sum_{k=\ell+1}^{n} \sum_{p=\ell+1}^{n} \left[\sum_{j=1}^{\ell} \frac{(a_{v_{\ell+1}} - a_{v_j})^2}{\rho^{2\delta}_{v_k v_j} \rho^{2\delta}_{v_p v_j}} \right]^{\frac{1}{2}} P'_{v_k v_\ell} \right\}. \tag{59}$$

Now, under the total power constraint (57), we can show that $\mathbb{E}C_{\text{T}}$ is upper bounded by the total power P_{total} to within a constant factor if the coefficients of all the $P_{u_k u_\ell}$ and $P'_{v_k v_\ell}$ in (59) are uniformly bounded by a constant. That is, we need to show that the following terms are uniformly bounded:

$$\sum_{p=1}^{\ell-1} \left[\sum_{j=\ell}^{n} \frac{(a_{u_j} - a_{u_{\ell-1}})^2}{\rho^{2\delta}_{u_k u_j} \rho^{2\delta}_{u_p u_j}} \right]^{\frac{1}{2}},\ \forall 1 \leq k < \ell \leq n, \tag{60}$$

$$\sum_{p=\ell+1}^{n} \left[\sum_{j=1}^{\ell} \frac{(a_{u_{\ell+1}} - a_{u_j})^2}{\rho^{2\delta}_{u_k u_j} \rho^{2\delta}_{u_p u_j}} \right]^{\frac{1}{2}},\ \forall 1 \leq \ell < k \leq n, \tag{61}$$

$$\sum_{p=1}^{\ell-1} \left[\sum_{j=\ell}^{n} \frac{(a_{v_j} - a_{v_{\ell-1}})^2}{\rho^{2\delta}_{v_k v_j} \rho^{2\delta}_{v_p v_j}} \right]^{\frac{1}{2}},\ \forall 1 \leq k < \ell \leq n, \tag{62}$$

$$\sum_{p=\ell+1}^{n} \left[\sum_{j=1}^{\ell} \frac{(a_{v_{\ell+1}} - a_{v_j})^2}{\rho^{2\delta}_{v_k v_j} \rho^{2\delta}_{v_p v_j}} \right]^{\frac{1}{2}},\ \forall 1 \leq \ell < k \leq n. \tag{63}$$

Towards this end, we will use the following bound: For any node $i \in \mathcal{N}$, $\rho_0 \geq \rho_{\min}$ and $\delta > 2$

$$\sum_{\substack{j\in\mathcal{N} \\ \rho_{ij} \geq \rho_0}} \frac{1}{\rho^{\delta}_{ij}} \overset{(a)}{\leq} \sum_{\substack{j\in\mathcal{N} \\ \rho_{ij} \geq \rho_0}} \frac{16}{\pi \rho^2_{\min}} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{0}^{\frac{\rho_{\min}}{2}} \frac{1}{(\rho^2_{ij} + r^2 - 2r\rho_{ij}\cos\theta)^{\frac{\delta}{2}}} r dr d\theta$$
$$\overset{(b)}{\leq} \frac{16}{\pi \rho^2_{\min}} \int_{-\pi}^{\pi} \int_{\rho_0 - \frac{\rho_{\min}}{2}}^{\infty} \frac{1}{\rho^{\delta}} \rho d\rho d\theta$$
$$= \frac{32}{(\delta - 2)\rho^2_{\min}} \frac{1}{\left(\rho_0 - \frac{\rho_{\min}}{2}\right)^{\delta-2}}, \tag{64}$$

where the inequality (a) follows from the observation[2] that for any $\theta \in [-\frac{\pi}{4}, \frac{\pi}{4}]$ and $0 \leq r \leq \frac{\rho_{\min}}{2}$,

$$\frac{1}{\rho^{\delta}_{ij}} \leq \frac{1}{(\rho^2_{ij} + r^2 - 2r\rho_{ij}\cos\theta)^{\frac{\delta}{2}}},$$

and the inequality (b) follows from the observation that all the open disks centered at node $j \in \mathcal{N}$ with radius $\frac{\rho_{\min}}{2}$ are disjoint with each other.

Let $\mathbf{1}_{\{A\}}$ be the indicator function of the event A (defined to be one if A is true and zero otherwise). For $\delta > 2$, for any $1 \leq k \neq p < \ell \leq n$, we have

$$\sum_{j=\ell}^{n} (a_{u_j} - a_{u_{\ell-1}})^2 \frac{1}{\rho^{2\delta}_{u_k u_j}} \frac{1}{\rho^{2\delta}_{u_p u_j}} \mathbf{1}_{\{\rho_{u_k u_j} \leq \rho_{u_p u_j}\}}$$
$$\overset{(a)}{\leq} \sum_{j=\ell}^{n} \frac{1}{\rho^{2\delta-2}_{u_k u_j}} \frac{1}{\rho^{2\delta}_{u_p u_j}} \mathbf{1}_{\{\rho_{u_k u_j} \leq \rho_{u_p u_j}\}}$$
$$= \sum_{j=\ell}^{n} \frac{1}{\rho^{2\delta-2}_{u_k u_j}} \frac{1}{\rho^{2\delta}_{u_p u_j}} \mathbf{1}_{\{\rho_{u_k u_j} \leq \rho_{u_p u_j},\, \rho_{u_k u_j} \geq \rho_{u_k u_p}\}}$$
$$+ \sum_{j=\ell}^{n} \frac{1}{\rho^{2\delta-2}_{u_k u_j}} \frac{1}{\rho^{2\delta}_{u_p u_j}} \mathbf{1}_{\{\rho_{u_k u_j} \leq \rho_{u_p u_j},\, \rho_{u_k u_j} < \rho_{u_k u_p}\}}$$
$$\leq \sum_{j=\ell}^{n} \frac{1}{\rho^{4\delta-2}_{u_k u_j}} \mathbf{1}_{\{\rho_{u_k u_j} \geq \rho_{u_k u_p}\}} + \sum_{j=\ell}^{n} \frac{1}{\rho^{2\delta-2}_{u_k u_j}} \frac{1}{\left(\rho_{u_k u_p}/2\right)^{2\delta}}$$

[2]Consider a triangle with ρ_{ij} and r as the lengths of two sides, with an angle θ between them. Then the third side has a length of $(\rho^2_{ij} + r^2 - 2r\rho_{ij}\cos\theta)^{1/2}$, by the triangle formula. When $\rho_{ij} \geq \rho_{\min}$, $\theta \in [-\frac{\pi}{4}, \frac{\pi}{4}]$ and $0 \leq r \leq \frac{\rho_{\min}}{2}$, the length of the third side is no more than ρ_{ij}.

$$\overset{(b)}{\leq} \frac{32}{(4\delta-4)\rho_{\min}^2}\frac{1}{\left(\rho_{u_ku_p}-\frac{\rho_{\min}}{2}\right)^{4\delta-4}} + \frac{32}{(2\delta-4)\rho_{\min}^2}\frac{1}{\left(\rho_{\min}-\frac{\rho_{\min}}{2}\right)^{2\delta-4}}\frac{1}{\left(\rho_{u_ku_p}/2\right)^{2\delta}}$$
$$= \frac{8}{(\delta-1)\rho_{\min}^2}\frac{1}{\left(\rho_{u_ku_p}-\frac{\rho_{\min}}{2}\right)^{4\delta-4}} + \frac{2^{4\delta}}{(\delta-2)\rho_{\min}^{2\delta-2}}\frac{1}{\rho_{u_ku_p}^{2\delta}}$$

where the inequality (a) follows since $(a_{u_j}-a_{u_{\ell-1}}) \leq (a_{u_j}-a_{u_k}) \leq \rho_{u_ku_j}$, and the inequality (b) follows by the bound (64). Similarly, we can prove the same bound for

$$\sum_{j=\ell}^{n}(a_{u_j}-a_{u_{\ell-1}})^2\frac{1}{\rho_{u_ku_j}^{2\delta}}\frac{1}{\rho_{u_pu_j}^{2\delta}}\mathbf{1}_{\{\rho_{u_ku_j}>\rho_{u_pu_j}\}}.$$

Hence, we have

$$\sum_{j=\ell}^{n}(a_{u_j}-a_{u_{\ell-1}})^2\frac{1}{\rho_{u_ku_j}^{2\delta}}\frac{1}{\rho_{u_pu_j}^{2\delta}} \leq \frac{16}{(\delta-1)\rho_{\min}^2}\frac{1}{\left(\rho_{u_ku_p}-\frac{\rho_{\min}}{2}\right)^{4\delta-4}} + \frac{2^{4\delta+1}}{(\delta-2)\rho_{\min}^{2\delta-2}}\frac{1}{\rho_{u_ku_p}^{2\delta}}.$$

Therefore, for $\delta > 2$, we have the following uniform upper bound for all the terms in (60):

$$\max_{1\leq k<\ell\leq n}\sum_{p=1}^{\ell-1}\left[\sum_{j=\ell}^{n}(a_{u_j}-a_{u_{\ell-1}})^2\frac{1}{\rho_{u_ku_j}^{2\delta}}\frac{1}{\rho_{u_pu_j}^{2\delta}}\right]^{\frac{1}{2}}$$
$$= \max_{1\leq k<\ell\leq n}\left\{\sum_{\substack{1\leq p\leq \ell-1\\ p\neq k}}\left[\sum_{j=\ell}^{n}(a_{u_j}-a_{u_{\ell-1}})^2\frac{1}{\rho_{u_ku_j}^{2\delta}}\frac{1}{\rho_{u_pu_j}^{2\delta}}\right]^{\frac{1}{2}} + \left[\sum_{j=\ell}^{n}(a_{u_j}-a_{u_{\ell-1}})^2\frac{1}{\rho_{u_ku_j}^{2\delta}}\frac{1}{\rho_{u_ku_j}^{2\delta}}\right]^{\frac{1}{2}}\right\}$$
$$\leq \max_{1\leq k<\ell\leq n}\sum_{\substack{1\leq p\leq \ell-1\\ p\neq k}}\left\{\left[\frac{16}{(\delta-1)\rho_{\min}^2}\frac{1}{\left(\rho_{u_ku_p}-\frac{\rho_{\min}}{2}\right)^{4\delta-4}}\right]^{\frac{1}{2}} + \left[\frac{2^{4\delta+1}}{(\delta-2)\rho_{\min}^{2\delta-2}}\frac{1}{\rho_{u_ku_p}^{2\delta}}\right]^{\frac{1}{2}}\right\} + \max_{1\leq k<\ell\leq n}\left[\sum_{j=\ell}^{n}\frac{1}{\rho_{u_ku_j}^{4\delta-2}}\right]^{\frac{1}{2}}$$
$$= \max_{1\leq k<\ell\leq n}\left\{\frac{4}{(\delta-1)^{\frac{1}{2}}\rho_{\min}}\sum_{\substack{1\leq p\leq \ell-1\\ p\neq k}}\frac{1}{\left(\rho_{u_ku_p}-\frac{\rho_{\min}}{2}\right)^{2\delta-2}} + \frac{2^{2\delta+\frac{1}{2}}}{(\delta-2)^{\frac{1}{2}}\rho_{\min}^{\delta-1}}\sum_{\substack{1\leq p\leq \ell-1\\ p\neq k}}\frac{1}{\rho_{u_ku_p}^{\delta}}\right\} + \max_{1\leq k<\ell\leq n}\left[\sum_{j=\ell}^{n}\frac{1}{\rho_{u_ku_j}^{4\delta-2}}\right]^{\frac{1}{2}}$$
$$\overset{(a)}{\leq} \frac{4}{(\delta-1)^{\frac{1}{2}}\rho_{\min}}\frac{128}{\rho_{\min}^2}\left[\frac{1}{2\delta-4}\frac{1}{(\frac{\rho_{\min}}{4})^{2\delta-4}} + \frac{1}{2\delta-3}\times\frac{\frac{\rho_{\min}}{2}}{(\frac{\rho_{\min}}{4})^{2\delta-3}}\right] + \frac{2^{2\delta+\frac{1}{2}}}{(\delta-2)^{\frac{1}{2}}\rho_{\min}^{\delta-1}}\frac{32}{(\delta-2)\rho_{\min}^2}\frac{1}{\left(\rho_{\min}-\frac{\rho_{\min}}{2}\right)^{\delta-2}} + \left[\frac{32}{(4\delta-4)\rho_{\min}^2}\frac{1}{\left(\rho_{\min}-\frac{\rho_{\min}}{2}\right)^{4\delta-4}}\right]^{\frac{1}{2}}$$
$$= \left[\frac{2^{4\delta}}{(\delta-1)^{\frac{1}{2}}(\delta-2)} + \frac{2^{4\delta+2}}{(\delta-1)^{\frac{1}{2}}(2\delta-3)} + \frac{2^{3\delta+\frac{7}{2}}}{(\delta-2)^{\frac{3}{2}}} + \frac{2^{2\delta-\frac{1}{2}}}{(\delta-1)^{\frac{1}{2}}}\right]\frac{1}{\rho_{\min}^{2\delta-1}}$$

where the inequality (a) follows from the bound (64) and the following bound:

$$\sum_{\substack{1\leq p\leq \ell-1\\ p\neq k}}\frac{1}{\left(\rho_{u_ku_p}-\frac{\rho_{\min}}{2}\right)^{2\delta-2}}$$
$$\leq \sum_{\substack{1\leq p\leq \ell-1\\ p\neq k}}\frac{64}{\pi\rho_{\min}^2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_0^{\frac{\rho_{\min}}{4}}\frac{1}{[(\rho_{u_ku_p}^2+r^2-2r\rho_{u_ku_p}\cos\theta)^{\frac{1}{2}}-\frac{\rho_{\min}}{2}]^{2\delta-2}}rdrd\theta$$
$$\leq \frac{64}{\pi\rho_{\min}^2}\int_{-\pi}^{\pi}\int_{\rho_{\min}-\frac{\rho_{\min}}{4}}^{\infty}\frac{1}{(\rho-\frac{\rho_{\min}}{2})^{2\delta-2}}\rho d\rho d\theta$$
$$= \frac{128}{\rho_{\min}^2}\left[\frac{1}{2\delta-4}\frac{1}{(\frac{\rho_{\min}}{4})^{2\delta-4}} + \frac{1}{2\delta-3}\frac{\frac{\rho_{\min}}{2}}{(\frac{\rho_{\min}}{4})^{2\delta-3}}\right],$$

which follows similarly as (64), but with smaller disks of radius $\frac{\rho_{\min}}{4}$.

Similarly, we can prove that the same upper bound holds for all the terms in (61)-(63). This leads to the following theorem.

Theorem 5.2: Let the phases $\{\theta_{ij}\}$ be independent of each other, and uniformly distributed on $[0, 2\pi)$, but with their realizations known to all the nodes, i.e., all channel state informations (CSI) are known to all transmitters and all receivers. Under the total power constraint, for $\delta > 2$, the expected transport capacity is always upper bounded by the total power to within a constant factor:

$$\mathbb{E}\, C_{\mathrm{T}} \leq \frac{\bar{c}_2(\delta)\log e}{\sigma^2\rho_{\min}^{2\delta-1}}P_{\text{total}} \tag{65}$$

where

$$\bar{c}_2(\delta) = \frac{2^{4\delta+2}}{(\delta-1)^{\frac{1}{2}}(\delta-2)} + \frac{2^{4\delta+4}}{(\delta-1)^{\frac{1}{2}}(2\delta-3)} + \frac{2^{3\delta+\frac{11}{2}}}{(\delta-2)^{\frac{3}{2}}} + \frac{2^{2\delta+\frac{3}{2}}}{(\delta-1)^{\frac{1}{2}}}. \tag{66}$$

The following theorem establishing a linear scaling law under the individual power constraint follows immediately by noting that $P_{\text{total}} = n \cdot P_{\text{ind}}$.

Theorem 5.3: Let the phases $\{\theta_{ij}\}$ be independent of each other and uniformly distributed on $[0, 2\pi)$, but with their realizations known to all the nodes, i.e., the information states of all channels are known to all nodes. Under the individual power constraint, for $\delta > 2$, the expected transport capacity is always upper bounded by the number of nodes n to within a constant factor:

$$\mathbb{E}\, C_{\mathrm{T}} \leq \frac{\bar{c}_2(\delta)\log e}{\sigma^2\rho_{\min}^{2\delta-1}}P_{\text{ind}}\cdot n \tag{67}$$

where $\bar{c}_2$ is defined in (66).

B. Uniform Upper Bounds Irrespective of Phases

In this section, we develop uniform upper bounds on the transport capacity for all possible realizations of the phases $\{\theta_{ij}\}$.

The following theorem follows immediately from the bound (56).

Theorem 5.4: The transport capacity is upper bounded by

$$C_{\mathrm{T}} \leq \max_{\{P_{u_k u_\ell}\}} \frac{\log e}{\sigma^2} \sum_{i=1}^{n-1} (a_{u_{i+1}} - a_{u_i}) \left[\sum_{j=i+1}^{n} \sum_{\ell=2}^{i+1} \left(\sum_{k=1}^{\ell-1} \frac{\sqrt{P_{u_k u_\ell}}}{\rho_{u_k u_j}^{\delta}} \right)^2 + \sum_{j=1}^{i} \sum_{\ell=i}^{n-1} \left(\sum_{k=\ell+1}^{n} \frac{\sqrt{P_{u_k u_\ell}}}{\rho_{u_k u_j}^{\delta}} \right)^2 \right] + \max_{\{P'_{v_k v_\ell}\}} \frac{\log e}{\sigma^2} \sum_{i=1}^{n-1} (b_{v_{i+1}} - b_{v_i}) \left[\sum_{j=i+1}^{n} \sum_{\ell=2}^{i+1} \left(\sum_{k=1}^{\ell-1} \frac{\sqrt{P'_{v_k v_\ell}}}{\rho_{v_k v_j}^{\delta}} \right)^2 + \sum_{j=1}^{i} \sum_{\ell=i}^{n-1} \left(\sum_{k=\ell+1}^{n} \frac{\sqrt{P'_{v_k v_\ell}}}{\rho_{v_k v_j}^{\delta}} \right)^2 \right] \tag{68}$$

where the nonnegative powers $P_{u_k u_\ell}$ and $P'_{v_k v_\ell}$ ($1 \leq k, \ell \leq n$, and $k \neq \ell$) satisfy the total power constraint:

$$\begin{cases} \sum_{k=1}^{n} \max\left\{ \sum_{\ell=k+1}^{n} P_{u_k u_\ell} \,,\, \sum_{\ell=1}^{k-1} P_{u_k u_\ell} \right\} \leq P_{\text{total}} \\ \sum_{k=1}^{n} \max\left\{ \sum_{\ell=k+1}^{n} P'_{v_k v_\ell} \,,\, \sum_{\ell=1}^{k-1} P'_{v_k v_\ell} \right\} \leq P_{\text{total}} \end{cases} \tag{69}$$

or the individual power constraint: For all $k = 1, \ldots, n$,

$$\begin{cases} \max\left\{ \sum_{\ell=k+1}^{n} P_{u_k u_\ell} \,,\, \sum_{\ell=1}^{k-1} P_{u_k u_\ell} \right\} \leq P_{\text{ind}} \\ \max\left\{ \sum_{\ell=k+1}^{n} P'_{v_k v_\ell} \,,\, \sum_{\ell=1}^{k-1} P'_{v_k v_\ell} \right\} \leq P_{\text{ind}} \end{cases} \tag{70}$$

Similarly to the inequality (43), we have

$$\left(\sum_{k=1}^{\ell-1} \frac{\sqrt{P_{u_k u_\ell}}}{\rho_{u_k u_j}^{\delta}} \right)^2 \leq \sum_{k=1}^{\ell-1} \frac{1}{\rho_{u_k u_j}^{\delta}} \sum_{p=1}^{\ell-1} \frac{1}{\rho_{u_p u_j}^{\delta}} P_{u_k u_\ell},$$

$$\left(\sum_{k=\ell+1}^{n} \frac{\sqrt{P_{u_k u_\ell}}}{\rho_{u_k u_j}^{\delta}} \right)^2 \leq \sum_{k=\ell+1}^{n} \frac{1}{\rho_{u_k u_j}^{\delta}} \sum_{p=\ell+1}^{n} \frac{1}{\rho_{u_p u_j}^{\delta}} P_{u_k u_\ell},$$

$$\left(\sum_{k=1}^{\ell-1} \frac{\sqrt{P'_{v_k v_\ell}}}{\rho_{v_k v_j}^{\delta}} \right)^2 \leq \sum_{k=1}^{\ell-1} \frac{1}{\rho_{v_k v_j}^{\delta}} \sum_{p=1}^{\ell-1} \frac{1}{\rho_{v_p v_j}^{\delta}} P'_{v_k v_\ell},$$

$$\left(\sum_{k=\ell+1}^{n} \frac{\sqrt{P'_{v_k v_\ell}}}{\rho_{v_k v_j}^{\delta}} \right)^2 \leq \sum_{k=\ell+1}^{n} \frac{1}{\rho_{v_k v_j}^{\delta}} \sum_{p=\ell+1}^{n} \frac{1}{\rho_{v_p v_j}^{\delta}} P'_{v_k v_\ell}.$$

As in the one dimensional case, we can substitute the above inequalities into (68) so that C_{T} is upper bounded by the total power P_{total} to within a constant factor, if the coefficients of all the $P_{u_k u_\ell}$ and $P'_{v_k v_\ell}$ are uniformly bounded by a constant. That is, similarly to (45) and (46), we need to show that the following terms are uniformly bounded:

$$\sum_{j=\ell}^{n} \frac{(a_{u_j} - a_{u_{\ell-1}})}{\rho_{u_k u_j}^{\delta}} \sum_{p=1}^{\ell-1} \frac{1}{\rho_{u_p u_j}^{\delta}}, \quad \forall 1 \leq k < \ell \leq n, \tag{71}$$

$$\sum_{j=1}^{\ell} \frac{(a_{u_{\ell+1}} - a_{u_j})}{\rho_{u_k u_j}^{\delta}} \sum_{p=\ell+1}^{n} \frac{1}{\rho_{u_p u_j}^{\delta}}, \quad \forall 1 \leq \ell < k \leq n, \tag{72}$$

$$\sum_{j=\ell}^{n} \frac{(b_{v_j} - b_{v_{\ell-1}})}{\rho_{v_k v_j}^{\delta}} \sum_{p=1}^{\ell-1} \frac{1}{\rho_{v_p v_j}^{\delta}}, \quad \forall 1 \leq k < \ell \leq n, \tag{73}$$

$$\sum_{j=1}^{\ell} \frac{(b_{v_{\ell+1}} - b_{v_j})}{\rho_{v_k v_j}^{\delta}} \sum_{p=\ell+1}^{n} \frac{1}{\rho_{v_p v_j}^{\delta}}, \quad \forall 1 \leq \ell < k \leq n. \tag{74}$$

By the bound (64), for $\delta > 5/2$, we have the following uniform upper bound for all the terms in (71):

$$\begin{aligned} &\max_{1 \leq k < \ell \leq n} \sum_{j=\ell}^{n} \frac{(a_{u_j} - a_{u_{\ell-1}})}{\rho_{u_k u_j}^{\delta}} \sum_{p=1}^{\ell-1} \frac{1}{\rho_{u_p u_j}^{\delta}} \\ &\leq \max_{1 \leq k < \ell \leq n} \sum_{j=\ell}^{n} \frac{(a_{u_j} - a_{u_{\ell-1}})}{\rho_{u_k u_j}^{\delta}} \frac{32}{(\delta - 2)\rho_{\min}^2} \\ &\qquad \times \frac{1}{\left[(a_{u_j} - a_{u_{\ell-1}}) \vee \rho_{\min} - \frac{\rho_{\min}}{2} \right]^{\delta-2}} \\ &\leq \frac{32}{(\delta - 2)\rho_{\min}^2} \max_{1 \leq k < \ell \leq n} \left\{ \sum_{j=\ell}^{n} \frac{\left[(a_{u_j} - a_{u_{\ell-1}}) \vee \rho_{\min} - \frac{\rho_{\min}}{2} \right]^{3-\delta}}{\rho_{u_k u_j}^{\delta}} \right. \\ &\qquad \left. + \sum_{j=\ell}^{n} \frac{\frac{\rho_{\min}}{2}}{\rho_{u_k u_j}^{\delta}} \frac{1}{\left[(a_{u_j} - a_{u_{\ell-1}}) \vee \rho_{\min} - \frac{\rho_{\min}}{2} \right]^{\delta-2}} \right\} \end{aligned} \tag{75}$$

where for any $x, y \in \mathbb{R}^1$, $x \vee y := \max\{x, y\}$.

If $5/2 < \delta < 3$, then since $a_{u_j} - a_{u_{\ell-1}} \leq a_{u_j} - a_{u_k} \leq \rho_{u_k u_j}$ for any $k < \ell \leq j$, the R.H.S. of (75) is upper bounded by

$$\begin{aligned} &\frac{32}{(\delta - 2)\rho_{\min}^2} \max_{1 \leq k < \ell \leq n} \left\{ \sum_{j=\ell}^{n} \frac{\rho_{u_k u_j}^{3-\delta}}{\rho_{u_k u_j}^{\delta}} + \sum_{j=\ell}^{n} \frac{\frac{\rho_{\min}}{2}}{\rho_{u_k u_j}^{\delta}} \frac{1}{\left(\frac{\rho_{\min}}{2}\right)^{\delta-2}} \right\} \\ &= \frac{32}{(\delta - 2)\rho_{\min}^2} \max_{1 \leq k < \ell \leq n} \left\{ \sum_{j=\ell}^{n} \frac{1}{\rho_{u_k u_j}^{2\delta-3}} + \sum_{j=\ell}^{n} \frac{\left(\frac{\rho_{\min}}{2}\right)^{3-\delta}}{\rho_{u_k u_j}^{\delta}} \right\} \\ &\overset{(a)}{\leq} \left[\frac{32}{(\delta - 2)\rho_{\min}^2} \right]^2 \left\{ \frac{1}{\left[(a_{u_\ell} - a_{u_{\ell-1}}) \vee \rho_{\min} - \frac{\rho_{\min}}{2} \right]^{2\delta-5}} \right. \\ &\qquad \left. + \left(\frac{\rho_{\min}}{2} \right)^{3-\delta} \frac{1}{\left[(a_{u_\ell} - a_{u_{\ell-1}}) \vee \rho_{\min} - \frac{\rho_{\min}}{2} \right]^{\delta-2}} \right\} \\ &\leq \left[\frac{32}{(\delta - 2)\rho_{\min}^2} \right]^2 \left\{ \frac{1}{\left(\frac{\rho_{\min}}{2}\right)^{2\delta-5}} + \left(\frac{\rho_{\min}}{2} \right)^{3-\delta} \frac{1}{\left(\frac{\rho_{\min}}{2}\right)^{\delta-2}} \right\} \\ &= \frac{2^{2\delta+6}}{(\delta - 2)^2 \rho_{\min}^{2\delta-1}} \end{aligned}$$

where the inequality (a) follows from the bound (64).

If $\delta \geq 3$, then the R.H.S. of (75) is upper bounded by

$$\frac{32}{(\delta-2)\rho_{\min}^2} \max_{1\leq k<\ell\leq n} \left\{ \left(\frac{2}{\rho_{\min}}\right)^{\delta-3} \sum_{j=\ell}^{n} \frac{1}{\rho_{u_k u_j}^{\delta}} + \left(\frac{2}{\rho_{\min}}\right)^{\delta-3} \sum_{j=\ell}^{n} \frac{1}{\rho_{u_k u_j}^{\delta}} \right\}$$

$$\overset{(a)}{\leq} \left[\frac{32}{(\delta-2)\rho_{\min}^2}\right]^2 \frac{2^{\delta-2}}{\rho_{\min}^{\delta-3}} \frac{1}{\left[(a_{u_\ell}-a_{u_{\ell-1}}) \vee \rho_{\min} - \frac{\rho_{\min}}{2}\right]^{\delta-2}}$$

$$\leq \left[\frac{32}{(\delta-2)\rho_{\min}^2}\right]^2 \frac{2^{\delta-2}}{\rho_{\min}^{\delta-3}} \left(\frac{2}{\rho_{\min}}\right)^{\delta-2} = \frac{2^{2\delta+6}}{(\delta-2)^2\rho_{\min}^{2\delta-1}}$$

where the inequality (a) follows from the bound (64).

Similarly, we can prove that the same upper bound holds for all the terms in (72), (73) and (74).

This leads to the following theorem.

Theorem 5.5: Under the total power constraint, for $\delta > 5/2$, the transport capacity is always upper bounded by the total power to within a constant factor:

$$C_{\text{T}} \leq \frac{c_2(\delta)\log e}{\sigma^2\rho_{\min}^{2\delta-1}} P_{\text{total}} \tag{76}$$

where

$$c_2(\delta) := \frac{2^{2\delta+8}}{(\delta-2)^2}. \tag{77}$$

The following theorem establishing a linear scaling law under the individual power constraint follows immediately from $P_{\text{total}} = n \cdot P_{\text{ind}}$.

Theorem 5.6: Under the individual power constraint, for $\delta > 5/2$, the transport capacity is always upper bounded by the number of nodes n to a constant factor:

$$C_{\text{T}} \leq \frac{c_2(\delta)\log e}{\sigma^2\rho_{\min}^{2\delta-1}} P_{\text{ind}} \cdot n \tag{78}$$

where c_2 is defined in (77).

Next, we show that $\delta > 5/2$ is almost the weakest requirement for linear scaling law that we can prove with Theorem 5.1. That is, we will show that for any $\delta < 5/2$, under the individual power constraint (58), there exists a topology of the network such that the right-hand-side (R.H.S.) of (56) is not upper bounded by the number of nodes n to within any constant factor, i.e.,

$$\frac{\text{R.H.S. of (56)}}{n} \to \infty, \quad \text{as } n \to \infty. \tag{79}$$

Choose any $\rho_0 \geq \max\{\rho_{\min}, 1\}$. Consider regular planar networks with separation distance ρ_0, i.e., n nodes arranged with coordinates

$$(i\rho_0, j\rho_0) \quad \text{for } i=1,\ldots,m \text{ and } j=1,\ldots,m$$

where the integer m satisfies $m^2 \leq n < (m+1)^2$. The other $n-m^2$ nodes can be placed arbitrarily and they won't be counted in our calculations. Let $\theta_{ij} \equiv 0$, $\forall i \neq j$. For any two different nodes (i_1, j_1) and (i_2, j_2) (we denote the nodes by their coordinates), let

$$P_{(i_1,j_1),(i_2,j_2)} = \frac{P_{\text{ind}}}{m^2} \quad \text{and}$$
$$P'_{(i_1,j_1),(i_2,j_2)} = \phi_{(i_1,j_1),(i_2,j_2)} = \phi'_{(i_1,j_1),(i_2,j_2)} = 0.$$

It is easy to check that the individual power constraint (58) is satisfied. For this choice, the R.H.S. of (56) simplifies to

$$\frac{P_{\text{ind}}\log e}{m^2\sigma^2} \sum_{i=1}^{n-1} (a_{u_{i+1}} - a_{u_i}) \left[\sum_{j=i+1}^{n} \sum_{\ell=2}^{i+1} \left(\sum_{k=1}^{\ell-1} \frac{1}{\rho_{u_k u_j}^{\delta}} \right)^2 + \sum_{j=1}^{i} \sum_{\ell=i}^{n-1} \left(\sum_{k=\ell+1}^{n} \frac{1}{\rho_{u_k u_j}^{\delta}} \right)^2 \right]$$

$$= \frac{\rho_0 P_{\text{ind}}\log e}{m^2\sigma^2} \sum_{i=1}^{m-1} \left[\sum_{j=i+1}^{m} \sum_{j'=1}^{m} \sum_{\ell=2}^{i+1} \sum_{\ell'=1}^{m} \left(\sum_{k=1}^{\ell-1} \sum_{k'=1}^{m} \frac{1}{\rho_{(k,k'),(j,j')}^{\delta}} \right)^2 + \sum_{j=1}^{i} \sum_{j'=1}^{m} \sum_{\ell=i}^{n-1} \sum_{\ell'=1}^{m} \left(\sum_{k=\ell+1}^{n} \sum_{k'=1}^{m} \frac{1}{\rho_{(k,k'),(j,j')}^{\delta}} \right)^2 \right]$$

$$= \frac{P_{\text{ind}}\log e}{m^2\sigma^2\rho_0^{2\delta-1}} \sum_{i=1}^{m-1} \left[\sum_{j=i+1}^{m} \sum_{j'=1}^{m} \sum_{\ell=2}^{i+1} \sum_{\ell'=1}^{m} \left(\sum_{k=1}^{\ell-1} \sum_{k'=1}^{m} \frac{1}{[(k-j)^2+(k'-j')^2]^{\frac{\delta}{2}}} \right)^2 + \sum_{j=1}^{i} \sum_{j'=1}^{m} \sum_{\ell=i}^{n-1} \sum_{\ell'=1}^{m} \left(\sum_{k=\ell+1}^{n} \sum_{k'=1}^{m} \frac{1}{[(k-j)^2+(k'-j')^2]^{\frac{\delta}{2}}} \right)^2 \right].$$

Let

$$B(m,\delta) = \frac{1}{m^2\rho_0^{2\delta-1}} \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \sum_{j'=1}^{m} \sum_{\ell=2}^{i+1} \sum_{\ell'=1}^{m} \left(\sum_{k=1}^{\ell-1} \sum_{k'=1}^{m} \frac{1}{[(k-j)^2+(k'-j')^2]^{\frac{\delta}{2}}} \right)^2.$$

Since $\rho_0 \geq 1$, $B(m,\delta)$ is a decreasing function of δ. Therefore, we only need to show that for any $2 < \delta < 5/2$,

$$\frac{B(m,\delta)}{n} \to \infty, \quad \text{as } n \to \infty. \tag{80}$$

Similarly to the one dimensional case, straight-forward calculations lead directly to (80), as follows:

First,

$$B(m,\delta) \geq \frac{1}{m^2\rho_0^{2\delta-1}} \sum_{i=\lceil m/2\rceil}^{m-1} \sum_{j=i+1}^{m} \sum_{j'=1}^{m} \sum_{\ell=\lceil j/2\rceil+1}^{i+1} \sum_{\ell'=1}^{m} \left(\sum_{k=1}^{\ell-1} \sum_{k'=1}^{m} \frac{1}{[(k-j)^2+(k'-j')^2]^{\frac{\delta}{2}}} \right)^2$$

$$\geq \frac{1}{m^2\rho_0^{2\delta-1}} \sum_{i=\lceil m/2\rceil}^{m-1} \sum_{j=i+1}^{m} \sum_{j'=1}^{m} \sum_{\ell=\lceil j/2\rceil+1}^{i+1} \sum_{\ell'=1}^{m} \left(\int_0^m \int_0^{\ell-1} \frac{1}{[(x-j)^2+y^2]^{\frac{\delta}{2}}} \, dxdy \right)^2$$

$$= \frac{1}{\rho_0^{2\delta-1}} \sum_{i=\lceil m/2\rceil}^{m-1} \sum_{j=i+1}^{m} \sum_{\ell=\lceil j/2\rceil+1}^{i+1} \left(\int_0^m \int_0^{\ell-1} \frac{1}{[(x-j)^2+y^2]^{\frac{\delta}{2}}} \, dxdy \right)^2. \tag{81}$$

For any $\ell \le j \le m$,

$$\int_0^m \int_0^{\ell-1} \frac{1}{[(x-j)^2+y^2]^{\frac{\delta}{2}}}\, dxdy$$
$$\ge \int_0^{\pi/4} \int_{\frac{j-\ell+1}{\cos\theta}}^{\frac{j}{\cos\theta}} \frac{1}{\rho^\delta} \rho d\rho d\theta$$
$$\ge \int_0^{\pi/4} \int_{j-\ell+1}^{j} \frac{1}{(\rho'/\cos\theta)^{\delta-1}} \frac{1}{\cos\theta} d\rho' d\theta$$
$$\ge \frac{1}{(\sqrt{2})^{\delta-2}} \int_0^{\pi/4} \int_{j-\ell+1}^{j} \frac{1}{\rho^{\delta-1}} d\rho d\theta$$
$$= \frac{\pi}{2^{\delta/2+1}(\delta-2)} \left[\frac{1}{(j-\ell+1)^{\delta-2}} - \frac{1}{j^{\delta-2}} \right].$$

Then, since $j-(\ell-1) < j/2$ for any $\lceil j/2 \rceil + 1 \le \ell \le i+1$, we have

$$\sum_{\ell=\lceil j/2\rceil+1}^{i+1} \left[\int_0^m \int_0^{\ell-1} \frac{1}{[(x-j)^2+y^2]^{\frac{\delta}{2}}}\, dxdy \right]^2$$
$$\ge \left[\frac{\pi}{2^{\delta/2+1}(\delta-2)} \right]^2 \sum_{\ell=\lceil j/2\rceil+1}^{i+1} \left[\frac{1}{(j-\ell+1)^{\delta-2}} - \frac{1}{j^{\delta-2}} \right]^2$$
$$\overset{(a)}{\ge} \left[\frac{\pi}{2^{\delta/2+1}(\delta-2)} \right]^2 \left(1 - \frac{1}{2^{\delta-2}} \right)^2 \int_{x=\lceil j/2\rceil}^{i} \frac{1}{(j-x)^{2\delta-4}}\, dx$$
$$= \alpha_3(\delta) \left[(j-\lceil j/2\rceil)^{5-2\delta} - (j-i)^{5-2\delta} \right]$$

where the inequality (a) follows similarly to (53), and

$$\alpha_3(\delta) = \left[\frac{\pi}{2^{\delta/2+1}(\delta-2)} \right]^2 \left(1 - \frac{1}{2^{\delta-2}} \right)^2 \frac{1}{(5-2\delta)} > 0. \tag{82}$$

Again, similarly to (54)-(55), we have

$$\sum_{i=\lceil m/2\rceil}^{m-1} \sum_{j=i+1}^{m} \left[(j-\lceil j/2\rceil)^{5-2\delta} - (j-i)^{5-2\delta} \right]$$
$$\ge \alpha_4(\delta) m^{7-2\delta} + o(1)$$

where

$$\alpha_4(\delta) = \frac{1}{6-2\delta} \left\{ \frac{1}{2^{6-2\delta}} - \frac{1}{7-2\delta}\frac{1}{2^{6-2\delta}} - \frac{1}{7-2\delta}\frac{1}{2^{7-2\delta}} \right\} > 0.$$

Therefore, finally by (81), we obtain (80).

The above example shows that with only Theorem 5.1, under the individual power constraint, we cannot expect to prove that the transport capacity is always upper bounded by the number of nodes n to within any constant factor, for any $\delta < 5/2$.

As a fringe benefit, it also shows that only with Theorem 5.1, under the total power constraint, we cannot expect to prove that the transport capacity is always upper bounded by the total power to a constant factor for any $\delta < 5/2$.

VI. Concluding Remarks

The results in this paper establish that for a path loss $\frac{e^{i\theta}}{\rho^\delta}$, where θ is a random phase, the expected transport capacity is at the best $\Theta(n)$ when $\delta > 2$ for two-dimensional networks, or $\delta > \frac{5}{4}$ for one-dimensional networks, even if all nodes have full information on all channel state information such as all random phases. This also has implications for the ergodic transport capacity of wireless networks over fading channels. It is interesting to know how far these bounds on δ can be decreased. Furthermore, for any realization of the phases, the transport capacity is uniformly upper bounded by a multiple of the total of the transmissions powers of all the nodes when $\delta > \frac{5}{2}$ in the two-dimensional case, or $\delta > \frac{3}{2}$ in the one-dimensional case. But this cannot hold true if $\delta < \frac{3}{2}$ in the two-dimensional case, or $\delta < 1$ in the one-dimensional case, as demonstrated by the multiple relay networks constructed in [10]. What happens in the transition region in the interval $\frac{3}{2} \le \delta \le \frac{5}{2}$ for two-dimensional networks, or $1 \le \delta \le \frac{3}{2}$ for one-dimensional networks, is still open. These appear as fairly challenging issues given the present state of knowledge regarding upper bounds in network information theory. It is also useful to sharpen the pre-constants, since they specify, for example in the latter case, the energy cost to be irreducibly paid for a single bit-meter of transport in any wireless network, and thus a fundamental constant of much interest.

Appendix
Proof of (16)

The proof idea follows from [14, p.37], which also appeared in [13]. First,

$$\begin{aligned} \|\gamma_{XY}\|^2 \mathbb{E}\|X\|^2 \mathbb{E}\|Y\|^2 &= \|\mathbb{E}XY^*\|^2 \\ &= \|\mathbb{E}[X\mathbb{E}(Y^*|X)]\|^2 \\ &\le \mathbb{E}\|X\|^2 \mathbb{E}\|\mathbb{E}(Y^*|X)\|^2 \end{aligned}$$

where the last inequality follows from the Cauchy-Schwarz Inequality, and "=" holds if (X,Y) is circularly symmetric complex Gaussian distributed with zero mean. Hence,

$$\|\gamma_{XY}\|^2 \mathbb{E}\|Y\|^2 \le \mathbb{E}\|\mathbb{E}(Y|X)\|^2. \tag{83}$$

Furthermore,

$$\begin{aligned} &\mathbb{E}\|Y\|^2 = \mathbb{E}[\mathbb{E}(\|Y\|^2|X)] \\ &= \mathbb{E}\big\{\mathbb{E}[\|Y-\mathbb{E}(Y|X)+\mathbb{E}(Y|X)\|^2|X]\big\} \\ &= \mathbb{E}\big\{\mathbb{E}[\|Y-\mathbb{E}(Y|X)\|^2|X] + \|\mathbb{E}(Y|X)\|^2\big\} \\ &= \mathbb{E}\big\{\mathbb{E}[\|Y-\mathbb{E}(Y|X)\|^2|X]\big\} + \mathbb{E}\|\mathbb{E}(Y|X)\|^2. \end{aligned} \tag{84}$$

Therefore, by (83)-(84), we have

$$\mathbb{E}\big\{\mathbb{E}[\|Y-\mathbb{E}(Y|X)\|^2|X]\big\} \le (1-\|\gamma_{XY}\|^2)\mathbb{E}\|Y\|^2 \tag{85}$$

where "=" holds if (X,Y) is circularly symmetric complex Gaussian distributed with zero mean.

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