Robust indirect adaptive control of time-varying plants with unmodeled dynamics and disturbances*

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Abstract

We show that indirect pole-zero placement adaptive controllers are robust for systems with time-varying parameters as well as unmodeled dynamics and disturbances. We use a parameter estimator with projection. No special signal normalization is employed to ensure robustness.

The nominal system parameters need only be bounded, and their variations need only be small in an average sense. This allows them to vary slowly with time, as well as to take large jumps occasionally. The adaptive controllers can simultaneously also tolerate small unmodeled dynamics, as well as bounded disturbances, with no restriction on the magnitude of the bound.

Key Words: Adaptive Systems, Disturbances, Indirect Adaptive Control, Robustness, Robust Performance, Stability, Time-varying plants, Unmodeled Dynamics.

AMS (MOS) subject classifications: 93B55 Pole and zero placement problems, 93C10 Nonlinear, 93C40 Adaptive control systems, 93C41 Problems with incomplete information, 93C50 Time-dependent, 93C55 Discrete-time, 93D21 Adaptive and robust stabilization, 93D25 Input-output approaches, 93D30 Scalar and vector Lyapunov functions, 93E12 System identification, 93E99 Stochastic learning and adaptive control.

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1 Introduction

In [1], Astrom and Wittenmark have proposed the use of indirect, certainty-equivalent self-tuning controllers based on pole-zero placement for the servo-problem. Thus, if the goal is to enforce a response

$$A^*(q^{-1})y_k = q^{-d}B^*(q^{-1})r_k \tag{1}$$

to a command signal $\{r_k\}$, then one first estimates system polynomials $\widehat{A}_{k-1}(q^{-1}) := 1 + \sum_{i=1}^{p} \widehat{a}_{i,k-1}q^{-i}$ and $\widehat{B}_{k-1}(q^{-1}) := \sum_{i=0}^{\ell-1} \widehat{b}_{i+1,k-1}q^{-i}$ at each time instant k by fitting a model,

$$\widehat{A}_k(q^{-1})y_t = \widehat{B}_k(q^{-1})u_{t-d}$$

to the available data $\{y_t, u_{t-1} | 0 \le t \le k\}$, at time k. Then, the control input u_k is chosen such that,

$$\widehat{R}_k(q^{-1})\widehat{B}_k(q^{-1})u_k = -\widehat{S}_k(q^{-1})y_k + B^*(q^{-1})r_k, \tag{2}$$

where the polynomials $\widehat{R}_k$ and $\widehat{S}_k$ are obtained by solving the Diophantine equation,

$$\widehat{R}_k(q^{-1})\widehat{A}_k(q^{-1}) + q^{-d}\widehat{S}_k(q^{-1}) = A^*(q^{-1}) \tag{3}$$

at each time instant. To estimate the parameter vector $\widehat{\theta}_k := (-\widehat{a}_{1,k}, \ldots, -\widehat{a}_{p,k}, \widehat{b}_{1,k}, \ldots, \widehat{b}_{\ell,k})^T$, one may use a recursive algorithm of the form

$$\widehat{\theta}_k = \widehat{\theta}_{k-1} + \Gamma_{k-1}\phi_{k-1}(y_k - \phi_{k-1}^T\widehat{\theta}_{k-1}),$$

where

$$\phi_{k-1} = (y_{k-1}, \ldots, y_{k-p}, u_{k-d}, \ldots, u_{k-\ell-(d-1)})^T. \tag{4}$$

The choice of the "gain" matrix $\Gamma_k = \gamma_k I$ yields a gradient update law. Control schemes of this nature have proven popular in practice, and many successful implementations have been reported; see [2, 3, 4, 5]. Also, the model reference adaptive control method, the backbone of continuous-time adaptive control, is a "direct" version of such a pole-zero placement scheme.

Often, such adaptive control algorithms are employed to control plants which are subject to time-variations, for which there is consequently an ever present need to adapt to the changing system characteristics. It is therefore important to develop a theory of robustness for such adaptive control algorithms for their use in the face of such system variations, as well as in the presence of uncertainties such as unmodeled dynamics and disturbances.

In this paper we consider an indirect adaptive control law, as above, with a parameter estimation scheme employing "projection" to keep the parameter estimates confined to a compact convex set. We establish that this simple modification is powerful enough to provide robustness simultaneously with respect to system time-variations, small unmodeled dynamics, and bounded disturbances. In particular, no special signal normalization is used.

Some background on the robustness problem for adaptive systems is in order. Much attention has been devoted over the past decade to the robustness problem caused by unmodeled dynamics and bounded disturbances, and an excellent unification of work up till 1988 is provided by Ioannou and Sun [6]. Recently, in an important paper, Ydstie [7] has shown that just the simple mechanism of "projection," which confines the parameter estimates to a compact convex set, is sufficient for robustness with respect to bounded disturbances and some unmodeled dynamics. This paper is notable also for the introduction of a new proof technique involving a "switched signal." This work has been extended to continuous-time, while also enlarging the class of unmodeled dynamics, by Naik, Kumar and Ydstie [8]. The net effect of these investigations is to show that many of the modifications, proposed in the 1980's to prove robustness with respect to unmodeled dynamics, are not necessary. The original simple "projection" modification, established by Egardt [9] to be robust with respect to bounded disturbances, is also robust with respect to small unmodeled dynamics.

Comparatively less attention has been devoted to the problem of robustness with respect to time-variations. Solo [10] has established the boundedness of signals for a direct one-step ahead adaptive control scheme using an a-posteriori estimate based gradient update law with leakage, when the parameters are slowly varying, while Kreisselmeier [11] has done so for an indirect adaptive control scheme with projection, for slow-in-the-mean parameter variations, which allows occasional large parameter jumps. Tsakalis and Ioannou [12] have obtained similar results for continuous-time plants, while Guo [13] and Meyn and Guo [14] have done so for discrete-time stochastic systems. In these works, while the plant is allowed to vary with time, the effect of unmodeled dynamics is not considered. In [15], de Larminat and Raynaud consider such robustness for a fairly general indirect adaptive control law which uses two parallel estimators as well as a specially constructed "normalization" signal. Middleton and Goodwin [16] also incorporate a normalization signal, and additionally assume knowledge of the constant factor by which it overbounds the unmodeled dynamics. This constant is then used to set up a normalized dead-zone, for which they establish robustness to slow-in-the-mean parameter variations and small unmodeled dynamics. Giri et al. [17] prove the robustness of an adaptive *regulator* for a plant with small-in-the-mean parameter variations and unmodelled dynamics, using only the knowledge of the order of a nominal plant model. However, this is done using a complicated control law involving an identification – stabilization time splitting, utilizing a least squares based adaptation law that also employs a special normalization signal. Furthermore, arbitrarily large bounded disturbances cannot be handled, and the regulation objective is *not* achieved in the "ideal" case unless the algorithm is appropriately modified.

In all these works, the signals entering the adaptation law are normalizaed by a specially constructed normalization signal, first proposed by Praly [18]. The effect of such normalization is to ensure that the modelling error entering the adaptation law is bounded or small in the mean. Construction of such a normalization signal requires a priori system knowledge and involves extra computation. In addition to these practical considerations, it is of theoretical interest to see if normalization is really necessary for robustness.

The key point of this paper is that such normalization is not required to ensure the robustness of adaptive controllers to small unmodeled dynamics, bounded disturbances with arbitrarily large bound, and slow-in-the-mean parameter variations. We show that one obtains robust adaptive control by merely utilizing parameter projection, together with "extended regressor" normalization, without recourse to any other modifications. An indirect adaptive pole-zero placement controller is considered. The resulting necessity to cancel process zeros requires the nominal portion of the time-varying plant to be minimum phase at every instant. On the other hand, the problem of potential loss of estimated model controllability/stabilizability faced in adaptive pole placment is not an issue here. This itself results in a simpler adaptive control scheme than those in [15], [17], and [19], for instance. A result similar to ours has recently been reported in Wen [20], using a different proof technique developed earlier in Wen and Hill [21]. They consider a unit delay plant with the time variations restricted to be slow. The true nominal time varying parameter vector is assumed to lie in a convex compact set, which is assumed to have the property that the nominal system polynomials induced by any parameter vector in the set are uniformly coprime. This restrictive assumption is the consequence of choosing an indirect pole–placement controller design. A final point worth noting is that unlike the signal bounds derived in the present paper, those in Wen [20] are not uniform in that they depend on the system initial condition.

Our main results are the following:
(i) A certainty equivalent adaptive controller, using a gradient based parameter estimator with projection, and employing normalization based on an "extended regressor," ensures that all closed loop signals are bounded, when applied to a nominally minimum phase discrete time plant with slow-in-the-mean parameter variations, bounded disturbances, and small unmodelled dynamics (Theorem 1).
(ii) In the absence of unmodelled dynamics and disturbances, and in case the parameter variations asymptotically tend to zero, i.e., in the nominal case, the error in tracking a reference trajectory converges to zero (Theorem 2). When unmodelled dynamics as well as bounded disturbances are present, the mean-squared output prediction error is linear in the magnitude of the unmodelled dynamics, bounded disturbances, and average rate of parameter variations (Theorem 3). Thus the adaptive controller provides robust performance in addition to robust boundedness.

2 System Description

Consider a plant

$$y_k + \sum_{i=1}^{p} a_{i,k-1} y_{k-i} = \sum_{j=1}^{\ell} b_{j,k-1} u_{k-j+1-d} + v_k, \tag{5}$$

where u and y denote the input and output, while v represents the cumulative effect of unmodelled dynamics and disturbances. The coefficients $\{a_{i,k}, b_{i,k}\}$ may vary with time, and so the system is allowed to be time-varying. We wish to investigate the robustness of indirect adaptive control schemes for such systems which simultaneously consist of time-variations, unmodelled dynamics and disturbances.

Let us suppose that the goal of adaptive control is to generate a closed-loop response to a command signal r_k, which satisfies (1), where q^{-1} denotes the usual backward-shift operator, i.e., $q^{-1}z_k := z_{k-1}$. If the plant in (5) were time-invariant and "ideal" without any unmodelled dynamics or disturbances, i.e., given by $A(q^{-1})y_k = q^{-d}B(q^{-1})u_k$, then the control law,

$$R(q^{-1})B(q^{-1})u_k = -S(q^{-1})y_k + B^*(q^{-1})r_k,$$

where $R(q^{-1})$ and $S(q^{-1})$ satisfy the Diophantine equation,

$$A(q^{-1})R(q^{-1}) + q^{-d}S(q^{-1}) = A^*(q^{-1}), \tag{6}$$

results in the cancellation of all original process zeroes (hence necessitating some sort of minimum-phase condition), and a closed-loop transfer-function $\frac{q^{-d}B^*(q^{-1})}{A^*(q^{-1})}$ from r_k to y_k, as desired.

We will consider here a certainty-equivalent indirect adaptive control scheme, which first forms parameter estimates

$$\widehat{\theta}_k = (-\widehat{a}_{1,k}, \ldots, -\widehat{a}_{p,k}, \widehat{b}_{1,k}, \ldots, \widehat{b}_{\ell,k})^T, \tag{7}$$

at each time-step k, and then uses the resulting estimated polynomials $\widehat{A}_{k-1}(q^{-1}) := 1 + \sum_{i=1}^{p} \widehat{a}_{i,k-1} q^{-i}$ and $\widehat{B}_{k-1}(q^{-1}) := \sum_{i=0}^{\ell-1} \widehat{b}_{i+1,k-1} q^{-i}$ to solve the Diophantine equation,

$$\widehat{R}_k(q^{-1})\widehat{A}_k(q^{-1}) + q^{-d}\widehat{S}_k(q^{-1}) = A^*(q^{-1}), \tag{8}$$

for $\widehat{R}_k$ and $\widehat{S}_k$.[1] We will consider the minimum degree solution for $\widehat{R}_{k-1}$, so that it is monic, and of degree $(d-1)$. We note that such a solution can be found through a $(d-1)$-step long division of the polynomial A^* by $\hat{A}$. Then we apply the input u_k given by,

$$\widehat{R}_k(q^{-1})\widehat{B}_k(q^{-1})u_k = -\widehat{S}_k(q^{-1})y_k + B^*(q^{-1})r_k. \tag{9}$$

It is well known in adaptive control, see Egardt [9], that if one simply uses a pure gradient based parameter estimator, then the resulting adaptive system is destabilized by even a small bounded disturbance. It is therefore necessary to somehow modify the parameter estimator to secure robustness with respect to even just bounded disturbances.

We will consider a parameter update law using parameter projection. This modification, motivated by the seminal work of Egardt [9], simply projects the parameter estimate vector onto a compact convex set $\mathcal{C}$ at each time step k. The set $\mathcal{C}$ is chosen such that

$$((p+1) - \text{th component of } \theta) \geq b_{\min} > 0 \text{ for every } \theta \in \mathcal{C}.$$

(By thus ensuring that the estimated leading coefficient of $\widehat{B}_k(q^{-1})$ is positive, we also ensure that the control law (9) is well defined, since it does not involve division by zero). The parameter estimates are recursively specified by

$$\widehat{\theta}'_k = \widehat{\theta}_{k-1} + \frac{\mu \phi_{k-1} e_k}{1 + \|\psi_{k-1}\|^2}, \tag{10}$$

$$e_k := y_k - \phi_{k-1}^T \widehat{\theta}_{k-1}, \tag{11}$$

and

$$\widehat{\theta}_k = \text{ Proj }_{\mathcal{C}}[\widehat{\theta}'_k], \tag{12}$$

where the regressor ϕ is given by (4). Here $\text{ Proj }_{\mathcal{C}}[\cdot]$ denotes "orthogonal" projection onto the compact convex set $\mathcal{C}$, defined uniquely by $\text{ Proj }_{\mathcal{C}}[x] \in \mathcal{C}$, and $\| \text{ Proj }_{\mathcal{C}}[x] - x\| \leq \|y - x\|, \forall y \in \mathcal{C}$. The vector ψ_k is an "extension" of the regression vector ϕ_k defined in (4),

$$\psi_{k-1} := \left(y_{k-1}, \ldots, y_{k-p'-d-m'+1}, u_{k-1}, \ldots, u_{k-\ell-2d-n'+2}\right)^T, \tag{13}$$

where $p' := p, m' := 0, n' := 0$ if $d = 1$, and $p' := \max\{p, \deg(A^*)\}$, $m' := \max\{0, p-d\}$, $n' := \max\{0, p'-d+1\}$ if $d > 1$.[2] Above, the constant μ in the step-size is chosen such that $0 < \mu < 2$, and the algorithm is initialized with $\hat{\theta}_0 \in \mathcal{C}$. We shall refer to this as the *Parameter Estimator with Projection* (PEP).

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[1] Since we are dealing with time-varying polynomials in a shift-operator, we distinguish between the notations, $C_k(q^{-1})D_k(q^{-1}) := \sum_{i=1}^{\ell} \sum_{j=1}^{\ell} c_{i,k} d_{j,k} q^{-(i+j)}$ and $C_k(q^{-1}) \circ D_k(q^{-1}) := \sum_{i=1}^{\ell} \sum_{j=1}^{\ell} c_{i,k} d_{j,k-i} q^{-(i+j)}$, when multiplying $C_k(q^{-1}) = \sum_{i=1}^{\ell} c_{i,k} q^{-i}$ and $D_k(q^{-1}) = \sum_{j=1}^{\ell} d_{j,k} q^{-j}$. For a signal x_k, $C_k(q^{-1})x_k \equiv C_k(q^{-1}) \circ x_k$ denotes $\sum_{i=1}^{\ell} c_{i,k} x_{k-i}$, as is usual.

[2] When the delay is unity, i.e., $d = 1$, then $\psi = \phi$, i.e., the extended regressor is identical to the regressor.

Remarks: All the results of this paper can be extended to least-squares type parameter estimators, if the condition number of the covariance matrix is kept bounded, and also to "direct" pole-zero placement schemes. In fact, though we do not show it here, if one uses a direct one-step ahead adaptive control law, then robustness can be established easily through a similar analysis.

3 The Assumptions

Our goal is to analyze the behavior of these adaptive controllers when they are applied to plants of the form shown in (5), which consist simultaneously of time-variations, unmodelled dynamics, and disturbances. We shall make the following assumptions on the plant (5) and the reference model (1).

(A1) $\theta_k \in \mathcal{C}$ for all k, where $\mathcal{C}$ is a compact convex set, such that the $(p+1)$-th component of every vector in $\mathcal{C}$ is larger than or equal to $b_{\min} > 0$. Let K_θ be a constant which bounds $\|\theta\|$ for all $\theta \in \mathcal{C}$.

(A2) The zeros of $B_k(q^{-1}) = \sum_{i=1}^{\ell} b_{i,k} q^{-i+1} = 0$ lie in the open disk $|q|^2 < \sigma'' < 1$, for all k.

(A3) $\sum_{k=t+1}^{t+T} \|\theta_k - \theta_{k-1}\| \leq K_\delta + k_\delta T$, for all $t, T \geq 0$, for some constants $K_\delta, k_\delta > 0$. Here $\theta_k := (a_{1,k}, \ldots, a_{p,k}, b_{1,k}, \ldots, b_{\ell,k})^T$ is the time-varying parameter vector.

(A4) $v_k^2 \leq K_v m_{k-1} + k_v$, for all k, for some $K_v, k_v > 0$, where m_k satisfies a recursion,

$$m_k = \sigma m_{k-1} + K_y y_k^2 + K_u u_k^2 + K_3, \quad m_0 > 0, \tag{14}$$

and σ is as in (A2).

(A5) $A^*(q^{-1}) = 1 + \sum_1^{p'} a_i^* q^{-i}$ has all its zeros in the open disk $|q|^2 < \sigma''' < 1$.[3] For later use, define $\sigma' := \max\{\sigma'', \sigma'''\}$.

The assumptions (A1) and (A2) require that the time-varying parameter vector θ_k be bounded, have leading b_1-coefficient uniformly bounded away from zero, and be strictly minimum-phase, at every time-instant k. This latter restriction is necessitated by the requirement of cancelling all process zeroes. Assumption (A3) allows slow-in-the-mean parameter time-variations, and thus occasional jumps too. Assumption (A4) allows small unmodelled dynamics, and bounded disturbances with arbitrarily large bound. Finally, (A5) simply requires the reference model to be stable with a prescribed margin of stability.

Remark: It is worthwhile to note that assumption (A4) includes the case of an incorrect assumption on the delay, since m_{k-1} includes input terms upto time $k-1$. Furthermore, the class of unmodelled dynamics covered by (A4) allows the true plant to be non-minimum phase [22, 23].

[3] Without loss of generality, if $\deg(A^*) < p'$, we set $a_i^* = 0$ for $i = \deg(A^*) + 1, \ldots, p'$.

In order to implement the above adaptive control schemes, one thus needs to know upper bounds p and ℓ on the orders of the *nominal* time-varying portion of the plant, the nominal delay d, and the sign (say, positive) and a lower bound $b_{\min} > 0$ on the leading term in the polynomial $B_k(q^{-1})$ modeling the numerator of the nominal portion of the plant, for every k. In addition, since one has to project the parameter estimates onto a compact convex set guaranteed to contain the true parameter θ_k for all k, one has to know the bound K_θ on the norms of the time-varying parameters $\|\theta_k\|$ for all k. Finally, one also needs to know the reference model given by $A^*(q^{-1})$, $B^*(q^{-1})$, and r_k.

Our central result in this paper is that under the above assumptions, the adaptive control laws are stable for all K_v and k_δ small enough, i.e., whenever the unmodelled dynamics is small enough, the parameter variations are small enough on the average, and the disturbances are bounded, though without any restriction on the magnitude of the bound.

4 Properties of the Parameter Estimator

We now derive some important properties of the parameter estimator PEP, which are independent of the control laws used. Let us denote by $\widetilde{\theta}_k := \widehat{\theta}_k - \theta_k$, the parameter estimation error. Using the definitions of ϕ_k and θ_k in (4) and (A3), we can express the plant (5) as

$$y_k = \phi_{k-1}^T \theta_{k-1} + v_k. \tag{15}$$

Further, substituting (15) in (11) gives

$$e_k = -\phi_{k-1}^T \widetilde{\theta}_{k-1} + v_k. \tag{16}$$

Lemma 1. *The following properties hold for the parameter estimator PEP.*

i) $\widehat{b}_{k,1} \geq b_{\min} > 0, \forall k \geq 0.$

ii) The parameter estimation errors are uniformly bounded, i.e., $\|\widetilde{\theta}_k\| \leq K_{\widetilde{\theta}}$ *for all* k, *where*

$$K_{\widetilde{\theta}} := 2K_\theta.$$

iii) The parameter estimates $\{\widehat{\theta}_k\}$ *are uniformly bounded also, with* $\|\widehat{\theta}_k\| \leq K_\theta$.

iv) Let ϵ *satisfy* $0 < \epsilon < 2 - \mu$. *Then,*

$$\sum_{k=t+1}^{t+T} \frac{e_k^2}{\rho_{k-1}} \leq K_{ev} \sum_{k=t+1}^{t+T} \frac{v_k^2}{\rho_{k-1}} + k_e T + K_e. \tag{17}$$

The quantity ρ_k *is defined as,*

$$\rho_{k-1} := 1 + \|\psi_{k-1}\|^2, \tag{18}$$

and the constants are given by, $K_{ev} := \frac{(1+\frac{1}{\epsilon})}{2-\mu-\epsilon}$, $k_e := \frac{2(K_\theta + K_{\widetilde{\theta}})(1+\mu)}{\mu(2-\mu-\epsilon)} k_\delta$, *and* $K_e := \frac{K_{\widetilde{\theta}}^2 + 2(K_\theta + K_{\widetilde{\theta}})(1+\mu) K_\delta}{\mu(2-\mu-\epsilon)}$.

Proof Define $\widetilde{\theta}'_k := \widehat{\theta}'_k - \theta_k$, $\overline{\phi}_{k-1} := \phi_{k-1}/\sqrt{\rho_{k-1}}$, $\bar{e}_k := e_k/\sqrt{\rho_{k-1}}$, and $\bar{v}_k := v_k/\sqrt{\rho_{k-1}}$.

i) This is immediate from the projection (12), and the property (A1) of $\mathcal{C}$.

ii) This is immediate from (A1) and the projection (12).

iii) This follows from (ii) by (A1).

iv) Let us define $\delta_k := \theta_k - \theta_{k-1}$. By (10),

$$\widetilde{\theta}'_k = \widetilde{\theta}_{k-1} - \delta_k + \mu\overline{\phi}_{k-1}\bar{e}_k. \tag{19}$$

Using $\|\overline{\phi}_{k-1}\|^2 \leq 1$ and (16), we get

$$2\mu\overline{\phi}_{k-1}^T\widetilde{\theta}_{k-1}\bar{e}_k = 2\mu(\bar{v}_k - \bar{e}_k)\bar{e}_k \leq \frac{\mu\bar{v}_k^2}{\epsilon} + \mu\epsilon\bar{e}_k^2 - 2\mu\bar{e}_k^2.$$

Using this inequality in (19) and noting that $\|\widetilde{\theta}'_k\| \leq \|\widetilde{\theta}_k\|$ due to projection, we get

$$\|\widetilde{\theta}_k\|^2 \leq \|\widetilde{\theta}_{k-1}\|^2 + \|\delta_k\|^2 + \mu^2\bar{e}_k^2 - 2\widetilde{\theta}_{k-1}^T\delta_k - 2\mu\overline{\phi}_{k-1}^T\delta_k\bar{e}_k + \frac{\mu\bar{v}_k^2}{\epsilon} + \mu\epsilon\bar{e}_k^2 - 2\mu\bar{e}_k^2. \tag{20}$$

Thus, using (16) we get,

$$\begin{aligned}
\|\delta_k\|^2 - 2\delta_k^T(\widetilde{\theta}_{k-1} + \mu\overline{\phi}_{k-1}\bar{e}_k) &\leq \|\delta_k\|^2 + 2\|\delta_k\|\left(\|\widetilde{\theta}_{k-1}\| + \mu|\bar{e}_k|\right) \\
&\leq \|\delta_k\|^2 + 2\|\delta_k\|\left(K_{\widetilde{\theta}} + \mu|\bar{v}_k - \overline{\phi}_{k-1}^T\widetilde{\theta}_{k-1}|\right) \\
&\leq \|\delta_k\|^2 + 2\|\delta_k\|\left(K_{\widetilde{\theta}} + \mu|\bar{v}_k| + \mu\|\widetilde{\theta}_{k-1}\|\right) \\
&\leq \|\delta_k\|^2 + 2\|\delta_k\|(1+\mu)K_{\widetilde{\theta}} + \mu(\|\delta_k\|^2 + \bar{v}_k^2) \\
&\leq \mu\bar{v}_k^2 + (1+\mu)\|\delta_k\|(\|\delta_k\| + 2K_{\widetilde{\theta}}) \\
&\leq \mu\bar{v}_k^2 + 2(1+\mu)\|\delta_k\|(K_\theta + K_{\widetilde{\theta}}).
\end{aligned}$$

Substituting this into inequality (20) gives

$$\begin{aligned}
\mu(2-\mu-\epsilon)\bar{e}_k^2 &\leq \|\widetilde{\theta}_{k-1}\|^2 - \|\widetilde{\theta}_k\|^2 + \|\delta_k\|^2 + \frac{\mu}{\epsilon}\bar{v}_k^2 - 2\delta_k^T(\widetilde{\theta}_{k-1} + \mu\overline{\phi}_{k-1}\bar{e}_k) \\
&\leq \|\widetilde{\theta}_{k-1}\|^2 - \|\widetilde{\theta}_k\|^2 + \mu\left(1 + \frac{1}{\epsilon}\right)\bar{v}_k^2 + 2(1+\mu)(K_\theta + K_{\widetilde{\theta}})\|\delta_k\|.
\end{aligned}$$

Summing from $t+1$ to $t+T$, telescoping, and using (A3) as well as $\|\widetilde{\theta}_k\|^2 \leq K_{\widetilde{\theta}}^2$, gives the desired result. □

The boundedness of parameter estimates yields the following important result that the controller parameters are Lipschitz functions of the plant parameters.

Lemma 2. *Let θ^c :=(coefficients of $R(q^{-1})$, coefficients of $S(q^{-1})$)^T denote the "controller" parameter vector, where R and S are functions of θ = (coefficients of $A(q^{-1};\theta)$, coefficients of $B(q^{-1};\theta)$)^T through (6). Let $\theta^{c,1}$ and $\theta^{c,2}$ denote the controller parameter vectors corresponding to two different plant parameter vectors θ^1 and θ^2 respectively. Then*

$$\|\theta^{c,1} - \theta^{c,2}\| \leq K(C)\|\theta^1 - \theta^2\|,$$

for all θ^1, $\theta^2 \in L_C := \{\theta \in R^{p+\ell} : \|\theta\| \leq C\}$, where $K(C)$ is a constant that only depends on C.

Proof The relation (6) can be re-written as a system of linear equations,

$$M(\theta^j)\theta^{c,j} = a^*, \ j = 1, 2,$$

where $M(\theta^j)$ denotes the Sylvester matrix corresponding to the polynomials $A(q^{-1})$ and q^{-d}, and a^* is formed from the coefficients of $A^*(q^{-1})$. Then, A being monic implies that there exists a positive ϵ such that, for all θ in L_C, $|\det(M(\theta^j))| \geq \epsilon$. We therefore get

$$\begin{aligned}
\|\theta^{c,1} - \theta^{c,2}\| &\leq \|M^{-1}(\theta^1) - M^{-1}(\theta^2)\| \cdot \|a^*\| \\
&\leq \|M^{-1}(\theta^1)\| \cdot \|M(\theta^1) - M(\theta^2)\| \cdot \|M^{-1}(\theta^2)\| \cdot \|a^*\| \\
&\leq \frac{(\|M(\theta^1)\| \cdot \|M(\theta^2)\|)^{p+\ell-1}}{|\det(M(\theta^1))\det(M(\theta^2))|}\|M(\theta^1) - M(\theta^2)\| \cdot \|a^*\| \\
&\leq K(C)\|M(\theta^1) - M(\theta^2)\| \\
&\leq \bar{K}(C)\|\theta^1 - \theta^2\|,
\end{aligned}$$

thereby concluding the proof. □

5 The Switched System

We now introduce, for purposes of analysis only, the "switched" system,

$$z_k = I_{k-1}(\sigma z_{k-1} + K_y e_k^2 + K_u u_{k-1}^2 + K_3) + (1 - I_{k-1})(g z_{k-1} + 2K_3); \quad z_0 > 0, \tag{21}$$

where $0 < \sigma < g < 1$, and the indicator function I_{k-1} is defined as,

$$I_{k-1} = \begin{cases} 1 & \text{if } \sigma z_{k-1} + K_y e_k^2 + K_u u_{k-1}^2 + K_3 \\ & \quad \geq g z_{k-1} + 2K_3, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3. *For all k,*

i) $\|\phi_k\|^2 \leq K m_k$ *and* $\|\psi_k\|^2 \leq K m_k$.[4]

ii) $m_k \leq K_{mz} z_k + k_{mz}$.

iii) $z_k \leq K_z z_{k-1} + k_z$.

iv) $v_k^2 \leq K_{vz} z_{k-1} + k_{vz}$.

v) $u_k^2 \leq K\rho_{k-1} + K_{uz} z_{k-1} + K$.

vi) $\rho_k \leq K\rho_{k-1} + K_{\rho z} z_{k-1} + K$.

vii) $\rho_k \leq K_{zk} + K$.

Above, the constants K_{vz}, K_{uz}, and $K_{\rho z}$ can all be made as small as desired by choosing K_v small enough.

[4] In the remainder of the paper, we will use the symbol K to generically denote any positive constant that does not depend on either K_v or k_δ, and whose exact value is unimportant for the proofs which follow.

Proof

i) These are obvious from (4), (13) and (14), where we also note that by choosing m_0 large, the constant K can be chosen independently of the initial conditions.

ii) From

$$m_k = \sigma^k m_0 + \sum_{j=0}^{k} \sigma^{k-j}[K_y y_j^2 + K_u u_j^2 + K_3],$$

$$z_k \geq \sigma^k z_0 + \sum_{j=0}^{k} \sigma^{k-j}[K_y e_j^2 + K_u u_{j-1}^2 + K_3],$$

it is clear that it suffices to show that,

$$\sum_{j=0}^{k} \sigma^{k-j} y_j^2 \leq K \sum_{j=0}^{k} \sigma^{k-j} e_j^2 + K \sum_{j=0}^{k} \sigma^{k-j} u_{j-1}^2 + K + K'\sigma^k, \quad (22)$$

$$\text{and} \sum_{j=0}^{k} \sigma^{k-j} u_j^2 \leq K \sum_{j=0}^{k} \sigma^{k-j} y_j^2 + K \sum_{j=0}^{k} \sigma^{k-j} u_{j-1}^2 + K + K'\sigma^k, \quad (23)$$

where K' denotes a constant that may depend on the initial conditions, since the effects of the initial conditions can then be accounted for by making z_0 appropriately large, thus ensuring that K_{mz} and k_{mz} do not depend on the initial conditions.

For simplicity, let us first consider the case $d = 1$. In that case $\widehat{R}_{k-1}(q^{-1}) = 1$, and $\widehat{S}_{k-1}(q^{-1}) = q(A^*(q^{-1}) - \widehat{A}_{k-1}(q^{-1}))$. Hence, dropping q^{-1} for brevity, and recalling the notation for multiplying time-varying polynomials in the shift operator, we obtain

$$\begin{aligned} A^* y_k &= (\widehat{R}_{k-1}\widehat{A}_{k-1} + q^{-1}\widehat{S}_{k-1}) y_k \\ &= \widehat{A}_{k-1} y_k - \widehat{R}_{k-1}\widehat{B}_{k-1} u_{k-1} + B^* r_{k-1} \text{ (from (9))} \\ &= q^{-1} B^* r_k + y_k - \phi_{k-1}^T \widehat{\theta}_{k-1} \text{ (since } \widehat{A}_{k-1} y_k - \widehat{B}_{k-1} u_{k-1} = y_k - \phi_{k-1}^T \widehat{\theta}_{k-1}) \\ &= q^{-1} B^* r_k + e_k. \end{aligned} \quad (24)$$

From (A5) it follows that there exist constants $0 < \gamma, \delta < 1$ such that $|q|^2 < \gamma^2 < \gamma^{1+\delta} < \sigma < 1$ for every root q of $A^*(q^{-1}) = 0$. Using the boundedness of $\{r_k\}$ it follows from (24) that

$$\sum_{j=0}^{k} \sigma^{k-j} y_j^2 \leq K + K'\sigma^k + K \sum_{i=0}^{k} \sigma^{k-i} e_i^2, \quad (25)$$

thus proving (22). For (23), recall from (9) that

$$u_k = \frac{1}{\widehat{b}_{1,k}} \left[- \sum_{j=2}^{\ell} \widehat{b}_{j,k} u_{k-j+1} - \sum_{j=0}^{\deg(\widehat{S})} \widehat{s}_{j,k} y_{k-j} + B^*(q^{-1}) r_k \right]. \quad (26)$$

Since $\widehat{b}_{1,k} > b_{\min} > 0$, $\widehat{\theta}_k$ is bounded, and the coefficients of $\widehat{S}_k(q^{-1})$ depend continuously on $\widehat{\theta}_k$, the result (23) follows from (26).

Now let us turn to the case $d > 1$. In that case,

$$\begin{aligned}
\widehat{A}_{k-1}y_k &= q^{-d}\widehat{B}_{k-1}u_k + (\widehat{A}_{k-1} - A_{k-1})y_k + q^{-d}(B_{k-1} - \widehat{B}_{k-1})u_k \\
&\quad + A_{k-1}y_k - q^{-d}B_{k-1}u_k \\
&= q^{-d}\widehat{B}_{k-1}u_k - \phi_{k-1}^T\widetilde{\theta}_{k-1} + v_k \\
&= \widehat{B}_{k-1}u_{k-d} + e_k.
\end{aligned}$$

Hence operating on both sides by $\widehat{R}_{k-1}$, and keeping in mind the consequences of multiplying time-varying polynomials in the shift-operator, and the associated notation, we have,

$$\begin{aligned}
\widehat{R}_{k-1}e_k &= \widehat{R}_{k-1} \circ e_k = \widehat{R}_{k-1} \circ [\widehat{A}_{k-1}y_k - \widehat{B}_{k-1}u_{k-d}] \\
&= \widehat{R}_{k-1}\widehat{A}_{k-1}y_k - \widehat{R}_{k-1}\widehat{B}_{k-1}u_{k-d} - (\widehat{R}_{k-1}\widehat{A}_{k-1} - \widehat{R}_{k-1} \circ \widehat{A}_{k-1})y_k \\
&\quad + (\widehat{R}_{k-1}\widehat{B}_{k-1} - \widehat{R}_{k-1} \circ \widehat{B}_{k-1})u_{k-d} \\
&= (A^* - q^{-d}\widehat{S}_{k-1})y_k - \widehat{R}_{k-d}\widehat{B}_{k-d}u_{k-d} + (\widehat{R}_{k-d}\widehat{B}_{k-d} - \widehat{R}_{k-1}\widehat{B}_{k-1})u_{k-d} \\
&\quad - (\widehat{R}_{k-1}\widehat{A}_{k-1} - \widehat{R}_{k-1} \circ \widehat{A}_{k-1})y_k + (\widehat{R}_{k-1}\widehat{B}_{k-1} - \widehat{R}_{k-1} \circ \widehat{B}_{k-1})u_{k-d} \\
&= A^*y_k + (\widehat{S}_{k-d} - \widehat{S}_{k-1})y_{k-d} - B^*r_{k-d} \\
&\quad + (\widehat{R}_{k-d}\widehat{B}_{k-d} - \widehat{R}_{k-1}\widehat{B}_{k-1})u_{k-d} - (\widehat{R}_{k-1}\widehat{A}_{k-1} - \widehat{R}_{k-1} \circ \widehat{A}_{k-1})y_k \\
&\quad + (\widehat{R}_{k-1}\widehat{B}_{k-1} - \widehat{R}_{k-1} \circ \widehat{B}_{k-1})u_{k-d} \text{ (using (9))}.
\end{aligned}$$

Hence,

$$\begin{aligned}
A^*y_k &= q^{-d}B^*r_k + \widehat{R}_{k-1}e_k + (\widehat{R}_{k-1}\widehat{A}_{k-1} - \widehat{R}_{k-1} \circ \widehat{A}_{k-1})y_k \\
&\quad + (\widehat{S}_{k-1} - \widehat{S}_{k-d})y_{k-d} - (\widehat{R}_{k-1}\widehat{B}_{k-1} - \widehat{R}_{k-1} \circ \widehat{B}_{k-1})u_{k-d} \qquad (27) \\
&\quad + (\widehat{R}_{k-1}\widehat{B}_{k-1} - \widehat{R}_{k-d}\widehat{B}_{k-d})u_{k-d}. \qquad (28)
\end{aligned}$$

Now note that

$$\begin{aligned}
|(& \widehat{R}_{k-1}\widehat{A}_{k-1} - \widehat{R}_{k-1} \circ \widehat{A}_{k-1})y_k - (\widehat{R}_{k-1}\widehat{B}_{k-1} - \widehat{R}_{k-1} \circ \widehat{B}_{k-1})u_{k-d} \\
+ & (\widehat{R}_{k-1}\widehat{B}_{k-1} - \widehat{R}_{k-d}\widehat{B}_{k-d})u_{k-d} + (\widehat{S}_{k-1} - \widehat{S}_{k-d})y_{k-d}| \\
\leq & |\sum_{i=1}^{d-1}\sum_{j=1}^{p}\widehat{r}_{i,k-1}(\widehat{a}_{j,k-1} - \widehat{a}_{j,k-i-1})y_{k-i-j}| \\
+ & |\sum_{i=1}^{d-1}\sum_{j=1}^{\ell}\widehat{r}_{i,k-1}(\widehat{b}_{j,k-1} - \widehat{b}_{j,k-i-1})u_{k-i-j-d+1}| \\
+ & |\sum_{i=1}^{d-1}(\widehat{R}_{k-i}\widehat{B}_{k-i} - \widehat{R}_{k-i-1}\widehat{B}_{k-i-1})u_{k-d}| + |\sum_{i=1}^{d-1}(\widehat{S}_{k-i} - \widehat{S}_{k-i-1})y_{k-d}| \\
\leq & K\sum_{i=1}^{d-1}\sum_{j=1}^{p}\sum_{n=1}^{i}|\widehat{a}_{j,k-n} - \widehat{a}_{j,k-n-1}||y_{k-i-j}| \\
+ & K\sum_{i=1}^{d-1}\sum_{j=1}^{\ell}\sum_{n=1}^{i}|\widehat{b}_{j,k-n} - \widehat{b}_{j,k-n-1}||u_{k-i-j-d+1}| \\
+ & |\sum_{i=1}^{d-1}(\widehat{R}_{k-i} - \widehat{R}_{k-i-1})\widehat{B}_{k-i}u_{k-d} + (\widehat{B}_{k-i} - \widehat{B}_{k-i-1})\widehat{R}_{k-i-1}u_{k-d}|
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \Big|\sum_{i=1}^{d-1}\sum_{j=0}^{\deg(\widehat{S})}(\widehat{s}_{j,k-i}-\widehat{s}_{j,k-i-1})y_{k-d-j}\Big| \\
&\leq \quad K\sum_{i=1}^{d-1}\sum_{j=1}^{p}\sum_{n=1}^{i}\|\widehat{\theta}_{k-n}-\widehat{\theta}_{k-n-1}\|\|\psi_{k-1-n}\| \\
&+ \quad K\sum_{i=1}^{d-1}\sum_{j=1}^{\ell}\sum_{n=1}^{i}\|\widehat{\theta}_{k-n}-\widehat{\theta}_{k-n-1}\|\|\psi_{k-1-n}\| \\
&+ \quad \Big|\sum_{i=1}^{d-1}\sum_{j=1}^{d-1}\sum_{m=1}^{\ell}(\widehat{r}_{j,k-i}-\widehat{r}_{j,k-i-1})\widehat{b}_{m,k-i}u_{k-d-j-m+1} \\
&+ \quad \sum_{i=1}^{d-1}\sum_{m=1}^{\ell}\sum_{j=1}^{d-1}(\widehat{b}_{m,k-i}-\widehat{b}_{m,k-i-1})\widehat{r}_{j,k-i-1}u_{k-d-j-m+1}\Big| \\
&+ \quad K\sum_{i=1}^{d-1}\sum_{j=0}^{\deg(\widehat{S})}\|\widehat{\theta}_{k-i}-\widehat{\theta}_{k-i-1}\||y_{k-d-j}| \\
&\leq \quad K\sum_{i=1}^{d-1}\sum_{j=1}^{p}\sum_{n=1}^{i}\|\widehat{\theta}'_{k-n}-\widehat{\theta}_{k-n-1}\|\|\psi_{k-1-n}\| \\
&+ \quad K\sum_{i=1}^{d-1}\sum_{j=1}^{\ell}\sum_{n=1}^{i}\|\widehat{\theta}'_{k-n}-\widehat{\theta}_{k-n-1}\|\|\psi_{k-1-n}\| \\
&+ \quad K\sum_{i=1}^{d-1}\sum_{j=1}^{d-1}\sum_{m=1}^{\ell}\|\widehat{\theta}'_{k-i}-\widehat{\theta}_{k-i-1}\|\|\psi_{k-i-1}\| + K\sum_{i=1}^{d-1}\sum_{j=0}^{\deg(\widehat{S})}\|\widehat{\theta}'_{k-i}-\widehat{\theta}_{k-i-1}\|\|\psi_{k-i-1}\| \\
&\leq \quad K\sum_{n=1}^{d-1}\|\widehat{\theta}'_{k-n}-\widehat{\theta}_{k-n-1}\|\|\psi_{k-1-n}\| \\
&\leq \quad K\sum_{n=1}^{d-1}|e_{k-n}|,
\end{aligned}
$$

where we have used Lemma 2. Now employing this bound in (27), we can establish (22) just as in the case of $d = 1$. The result (23) again follows from (22) by (26).

iii) From (21), $z_k \leq g z_{k-1} + K_y e_k^2 + K_u u_{k-1}^2 + 2K_3$. Thus we only need to bound e_k^2 and u_{k-1}^2 in terms of z_{k-1}. For the first, by (A4) and (i), it follows from (16), that

$$
\begin{aligned}
e_k^2 &\leq 2K_{\tilde{\theta}}^2\|\phi_{k-1}\|^2 + 2K_v m_{k-1} + 2k_v \leq 2(K_{\tilde{\theta}}^2 K + K_v)m_{k-1} + 2k_v \\
&\leq 2K_{mz}(K_{\tilde{\theta}}^2 K + K_v)z_{k-1} + 2(K_{\tilde{\theta}}^2 K + K_v)k_{mz} + 2k_v.
\end{aligned}
$$

Also, from (14), $u_{k-1}^2 \leq \frac{m_{k-1}}{K_u} \leq \frac{K_{mz}z_{k-1}+k_{mz}}{K_u}$.

iv) From (A4) and (ii), $v_k^2 \leq K_v(K_{mz}z_{k-1} + k_{mz}) + k_v$.

v) From the control law (26), and the boundedness of $\{r_k\}$, $u_k^2 \leq K[\sum_{i=1}^{\ell-1} u_{k-i}^2 + \sum_{i=0}^{\deg(\widehat{S})} y_{k-i}^2 + 1]$. Hence, $u_k^2 \leq K\rho_{k-1} + Ky_k^2 + K$. Since $y_k = \phi_{k-1}^T \theta_{k-1} + v_k$, we obtain

$$\begin{aligned} y_k^2 &\leq K\rho_{k-1} + 2v_k^2 \\ &\leq K\rho_{k-1} + KK_v z_{k-1} + K, \end{aligned} \tag{29}$$

and the required bound follows.

vi) Clearly, $\rho_k \leq K\rho_{k-1} + Ky_k^2 + Ku_k^2$. Hence, from (29) and (v), $\rho_k \leq K\rho_{k-1} + K(K_{uz} + K_{vz})z_{k-1} + K$.

vii) Using i), ii) of Lemma 3 gives $\|\psi_k\|^2 \leq Kz_k + K$. Hence $\rho_k \leq Kz_k + K$ for PEP. □

Now we examine the implications of the switching mechanism in more detail.

Lemma 4.

i) There exists a constant $\epsilon_{\rho z} > 0$, such that $\rho_k \geq \epsilon_{\rho z} z_k - k_{\rho z}$, whenever $I_k = 1$.

ii) For every positive integer N, there exist positive constants $L(N), \bar{k}(N)$, and $K_{v\max}(N)$ with the following property. If $K_v \in [0, K_{v\max}]$, $I_{t_1} = 1$, and $z_k \geq L$ for all $k \in [t_1 - N, t_1]$, then $\rho_k \geq \bar{k}(N) z_k$ for all $k \in [t_1 - N, t_1]$.

Proof

i) Suppose $I_{k-1} = 1$. Then, from the definition of I_{k-1}, $K_y e_k^2 + K_u u_{k-1}^2 \geq (g - \sigma) z_{k-1} + K_3$. Hence, we have

$$K_y e_k^2 + K_u \rho_{k-1} \geq (g - \sigma) z_{k-1} + K_3. \tag{30}$$

Now, if $\rho_{k-1} \geq \frac{(g-\sigma)z_{k-1}+K_3}{2K_u}$, then the claim is true with $\epsilon_{\rho z} > 0$ chosen smaller than $(g - \sigma)/2K_u$. So let us only consider the interesting case, $\rho_{k-1} \leq \frac{(g-\sigma)z_{k-1}+K_3}{2K_u}$. From (30) we then have $e_k^2 \geq \frac{(g-\sigma)z_{k-1}+K_3}{2K_y}$, while from (16), $e_k^2 \leq 2K_{\tilde{\theta}}^2 \|\phi_{k-1}\|^2 + 2v_k^2$, and so

$$2K_{\tilde{\theta}}^2 \|\phi_{k-1}\|^2 \geq e_k^2 - 2v_k^2 \geq \frac{(g - \sigma) z_{k-1} + K_3}{2K_y} - 2K_{vz} z_{k-1} - 2k_{vz}.$$

Since $\rho_{k-1} \geq \|\phi_{k-1}\|^2$, the required result follows by choosing K_{vz} small enough, and $\epsilon_{\rho z} < \frac{K_\phi}{2K_{\tilde{\theta}}^2}(\frac{(g-\sigma)}{2K_y} - 2K_{vz})$.

ii) We will bound the growth rate of ρ_k / z_k, and then use this in a reversed time argument. Consider $k \in [t_1 - N, t_1]$. Then

$$\frac{\rho_{k+1}}{z_{k+1}} \leq \frac{K\rho_k + K_{\rho z1} z_{k+1} + K_{\rho z} z_k + K}{z_{k+1}},$$

by Lemma 3(vi), where we set $K_{\rho z1} := 0$ in the case of PEP. Hence,

$$\begin{aligned} \frac{\rho_{k+1}}{z_{k+1}} &\leq \frac{K}{\sigma}\frac{\rho_k}{z_k} + (K_{\rho z1} + \frac{K_{\rho z}}{\sigma}) + \frac{K}{\sigma L} \\ &= K_a \frac{\rho_k}{z_k} + K_b, \text{ (since } z_{k+1} \geq \sigma z_k \text{ and } z_k \geq L), \end{aligned}$$

where K_a and K_b are appropriately chosen. So,

$$\begin{aligned}\frac{\rho_{t_1}}{z_{t_1}} &\leq K_a^{t_1-t}\left[\frac{\rho_t}{z_t}+\frac{K_b}{K_a-1}\right], \qquad \forall t\in[t_1-N,t_1] \\ &\leq K_a^{N}\left[\frac{\rho_t}{z_t}+\frac{K_b}{K_a-1}\right], \qquad \forall t\in[t_1-N,t_1].\end{aligned}$$

Hence,

$$\begin{aligned}\frac{\rho_t}{z_t} &\geq K_a^{-N}\frac{\rho_{t_1}}{z_{t_1}}-\frac{K_b}{K_a-1} \\ &\geq K_a^{-N}\left(\epsilon_{\rho z}-\frac{k_{\rho z}}{L}\right)-\frac{K_b}{K_a-1} \text{ (by (i) and since } z_{t_1}\geq L).\end{aligned}$$

Now we simply define

$$\bar{k}(N) := K_a^{-N}[\epsilon_{\rho z}-\frac{k_{\rho z}}{L}]-\frac{[K_{\rho z 1}+\frac{K_{\rho z}}{\sigma}+\frac{K}{\sigma L}]}{K_a-1},$$

noting that by choosing $L=L(N)$ large enough, and $K_{\rho z 1}, K_{\rho z}$ small enough (by choosing K_v small enough), we can ensure that $\bar{k}(N)>0$. □

6 A Representation of the Closed-Loop System

In this section, we obtain a non-minimal state space description of the closed-loop system, consisting of a stable state-transition matrix, and driven by a composite input consisting of the output prediction error, the unmodelled dynamics and disturbances, the filtered reference input, and in the case of $d>1$, also of "small" fractions of the input and output.

Consider first, for simplicity, the case $d=1$. Then, using (24),

$$\begin{aligned}B_{k-1}(q^{-1})u_{k-1} &= A_{k-1}(q^{-1})y_k-v_k \\ &= A_{k-1}(q^{-1})y_k-v_k-[A^*(q^{-1})y_k-B^*(q^{-1})r_{k-1}-e_k] \\ &= [A_{k-1}(q^{-1})-A^*(q^{-1})]y_k+e_k-v_k+B^*(q^{-1})r_{k-1}.\end{aligned}$$

Hence,

$$\begin{aligned}u_{k-1} &= \frac{1}{b_{1,k-1}}[A_{k-1}(q^{-1})-A^*(q^{-1})]y_k+\frac{e_k-v_k}{b_{1,k-1}} \\ &\quad +\frac{B^*(q^{-1})r_{k-1}}{b_{1,k-1}}-\frac{(b_{2,k-1}u_{k-2}+\ldots+b_{\ell,k-1}u_{k-\ell})}{b_{1,k-1}} \\ &= -b'_{2,k-1}u_{k-2}-\cdots-b'_{\ell,k-1}u_{k-\ell}+a'_{1,k-1}y_{k-1}+\ldots+a'_{p',k-1}y_{k-p'} \\ &\quad +\frac{e_k-v_k}{b_{1,k-1}}+r'_{k-1},\end{aligned}$$

where b'_{k-1}, a'_{k-1} and r'_{k-1} are defined appropriately. Hence, defining the "state-vector"

$$x_k=[y_k,y_{k-1},\ldots,y_{k-p'},u_{k-1},\ldots,u_{k-\ell+1}]^T,$$

we obtain the closed-loop system representation,

$$x_k = F_k x_{k-1} + b_{e,k} e_k + b_{v,k} v_k + b_{r,k} r'_{k-1} \tag{31}$$

where

$$\begin{aligned}
F_k &:= \begin{bmatrix} J & 0 \\ G_k & H_k \end{bmatrix} \\
J &= \begin{bmatrix} -a_1^* & \cdots & -a_{p'}^* \\ & & 0 \\ & \mathrm{I} & \vdots \\ & & 0 \end{bmatrix} \\
H_k &= \begin{bmatrix} -b'_{2,k-1} & \cdots & -b'_{\ell,k-1} \\ & & 0 \\ & \mathrm{I} & \vdots \\ & & 0 \end{bmatrix} \\
G_k &= \begin{bmatrix} a'_{1,k-1} & \cdots & a'_{p',k-1} \\ & & 0 \\ & 0 & \vdots \\ & & 0 \end{bmatrix} \\
b_{e,k} &= [1\ 0\ \cdots\ 0\ 1/b_{1,k-1}\ 0\ \cdots\ 0]^T \\
b_{v,k} &= [0\ \cdots 0\ -1/b_{1,k-1}\ 0\ \cdots\ 0]^T \\
b_{r,k} &= [b_{1,k-1}\ 0\ \cdots\ 0\ 1\ 0\ \cdots\ 0]^T.
\end{aligned}$$

We note, for future use, that the eigenvalues of F_k, at each instant k, are the roots of $A^*(q^{-1})$ and $B_{k-1}(q^{-1})$. Hence, by (A2) and (A5), F_k is a stable matrix with all eigenvalues λ_i lying in the disk $|\lambda_i|^2 < \sigma' < 1$.

Now let us turn to the case of $d > 1$. First, recall that (27) gives

$$A^* y_k = q^{-d} B^* r_k + \widehat{R}_{k-1} e_k + \Delta_{1,k} u_{k-d} + \Delta_{2,k} u_{k-d} + \Delta_{3,k} y_k + \Delta_{4,k} y_{k-d}, \tag{32}$$

where $\Delta_{1,k} := -(\widehat{R}_{k-1}\widehat{B}_{k-1} - \widehat{R}_{k-1} \circ \widehat{B}_{k-1})$, $\Delta_{2,k} := \widehat{R}_{k-1}\widehat{B}_{k-1} - \widehat{R}_{k-d}\widehat{B}_{k-d}$, $\Delta_{3,k} := \widehat{R}_{k-1}\widehat{A}_{k-1} - \widehat{R}_{k-1} \circ \widehat{A}_{k-1}$, and $\Delta_{4,k} := \widehat{S}_{k-1} - \widehat{S}_{k-d}$.

Next, operating by $\widehat{A}_k$ on (9), adding $q^{-d}\widehat{S}_k\widehat{B}_k u_k$ to both sides, and then using (8) gives

$$A^*\widehat{B}_k u_k = (\widehat{A}_k\widehat{R}_k\widehat{B}_k - \widehat{A}_k \circ (\widehat{R}_k\widehat{B}_k))u_k + q^{-d}\widehat{S}_k\widehat{B}_k u_k - \widehat{A}_k \circ \widehat{S}_k y_k + \widehat{A}_k B^* r_k.$$

This implies

$$\begin{aligned}
A^* B_k u_k &= -A^*\widetilde{B}_{k-1+d} u_k + A^*(\widetilde{B}_{k-1+d} - \widetilde{B}_k)u_k + (\widehat{A}_k\widehat{R}_k\widehat{B}_k - \widehat{A}_k \circ (\widehat{R}_k\widehat{B}_k))u_k \\
&\quad + \widehat{A}_k B^* r_k + (\widehat{A}_k\widehat{S}_k - \widehat{A}_k \circ \widehat{S}_k)y_k + \widehat{S}_k(-\widehat{A}_{k-1} y_k + q^{-d}\widehat{B}_{k-1} u_k) \\
&\quad + \widehat{S}_k(q^{-d}(\widehat{B}_k - \widehat{B}_{k-1})u_k - (\widehat{A}_k - \widehat{A}_{k-1})y_k),
\end{aligned} \tag{33}$$

where $\widetilde{B}_k := \widehat{B}_k - B_k$. Next, note that by (4), (7), (15), and (16),

$$\begin{aligned} -\widehat{A}_{k-1} y_k + q^{-d} \widehat{B}_{k-1} u_k &= -y_k + \phi_{k-1}^T \widehat{\theta}_{k-1} \\ &= -e_k, \end{aligned} \tag{34}$$

and that by (34) and (5),

$$\begin{aligned} -\widetilde{B}_{k-1+d} u_k &= \widetilde{A}_{k-1+d} y_{k+d} - \widetilde{B}_{k-1+d} u_k - \widetilde{A}_{k-1+d} y_{k+d} \\ &= e_{k+d} - v_{k+d} - \widetilde{A}_{k-1+d} y_{k+d}, \end{aligned} \tag{35}$$

where $\widetilde{A}_k := \widehat{A}_k - A_k$. Using (34) and (35) in (33), and defining $\Delta_{5,k} := A^*(\widetilde{B}_{k-1} - \widetilde{B}_{k-d})$, $\Delta_{6,k} := (\widehat{A}_{k-d} \widehat{R}_{k-d} \widehat{B}_{k-d} - \widehat{A}_{k-d} \circ (\widehat{R}_{k-d} \widehat{B}_{k-d}))$, $\Delta_{7,k} := (\widehat{A}_{k-d} \widehat{S}_{k-d} - \widehat{A}_{k-d} \circ \widehat{S}_{k-d})$, $\Delta_{8,k} := \widehat{S}_{k-d}(\widehat{B}_{k-d} - \widehat{B}_{k-d-1})$, and $\Delta_{9,k} := \widehat{S}_{k-d}(\widehat{A}_{k-d} - \widehat{A}_{k-d-1})$, we thus get,

$$\begin{aligned} A^* B_{k-d} u_{k-d} &= -A^* \widetilde{A}_{k-1} y_k + A^*(e_k - v_k) + \Delta_{5,k} u_{k-d} + \Delta_{6,k} u_{k-d} + \widehat{A}_{k-d} B^* r_{k-d} \\ &\quad + \Delta_{7,k} y_{k-d} - \widehat{S}_{k-d} e_{k-d} + \Delta_{8,k} u_{k-2d} - \Delta_{9,k} y_{k-d}. \end{aligned} \tag{36}$$

We can now use (36) and (32) to get the required closed-loop system representation. Define

$$\begin{aligned} x_k &:= [y_k, \ldots, y_{k-p'-\max(p,d)+1}, u_{k-d}, \ldots, u_{k-p'-\ell-d+1}]^T, \\ l_{1,k} &:= q^{-d} B^* r_k + \widehat{R}_{k-1} e_k + \Delta_{1,k} u_{k-d} + \Delta_{2,k} u_{k-d} + \Delta_{3,k} y_k + \Delta_{4,k} y_{k-d}, \\ l_{2,k} &:= A^*(e_k - v_k) + \Delta_{5,k} u_{k-d} + \Delta_{6,k} u_{k-d} + \widehat{A}_{k-d} B^* r_{k-d} \\ &\quad + \Delta_{7,k} y_{k-d} - \widehat{S}_{k-d} e_{k-d} + \Delta_{8,k} u_{k-2d} - \Delta_{9,k} y_{k-d}, \\ \text{and } C_k(q^{-1}) &:= A^*(q^{-1}) B_k(q^{-1}) \\ &=: b_{1,k}(1 + c'_{2,k} q^{-1} + \ldots + c'_{p'+\ell+1,k} q^{-(p'+\ell)}). \end{aligned}$$

Note that by assumptions (A2) and (A5), all roots of $C_k(q^{-1})$ lie in the open disk $|q|^2 < \sigma' < 1$, for all $k \geq 1$.

Using the above definitions, we get the representation,

$$x_k = F_k x_{k-1} + b_1 l_{1,k} + (b_2 / b_{1,k-d}) l_{2,k}, \tag{37}$$

where

$$\begin{aligned} F_k &:= \begin{bmatrix} J' & 0 \\ G'_k & H'_k \end{bmatrix}, \\ J' &= \begin{bmatrix} -a_1^* & \cdots & -a_{p'}^* & 0 & \cdots & 0 \\ & & & & & 0 \\ & & \mathrm{I} & & & \vdots \\ & & & & & 0 \end{bmatrix}, \\ G'_k &= \begin{bmatrix} \star & \cdots & \star \\ & & 0 \\ & 0 & \vdots \\ & & 0 \end{bmatrix}, \end{aligned}$$

$$
\begin{aligned}
H'_k &= \begin{bmatrix} -c'_{2,k-1} & \cdots & -c'_{p'+\ell+1,k-1} \\ & & 0 \\ & \mathrm{I} & \vdots \\ & & 0 \end{bmatrix}, \\
b_1 &= [1, 0, \ldots, 0]^T, \\
b_2 &= [0, \ldots, 0, 1, 0, \ldots, 0]^T,
\end{aligned}
$$

and the $\star$'s represent non-zero scalars whose exact values are unimportant. Note that the eigenvalues of F_k are the zeros of $A^*(q^{-1})$ and the zeros of $B_{k-1}(q^{-1})$. One difference from (31) is that for the case $d > 1$, the system (37) is driven by terms involving the input and output. However, these occur only in products with the Δ_i terms, which are either parameter or parameter estimate differences or "swapping" terms.

7 The Contraction Property

We will consider the "composite" Lyapunov function,

$$W_k := k_F x_k^T P_k x_k + z_k,$$

where $k_F > 0$, and $P_k = P_k^T > 0$ satisfies the discrete-time Lyapunov equation,

$$F_k^T P_k F_k - P_k = -I.$$

We note that since the F_k's lie in a compact set, and each F_k has all its eigenvalues inside the disk of radius $\sigma^{\frac{\prime 1}{2}}$, we have $\|F_k^n\| \le \epsilon_F \gamma^n \le \epsilon_F \sigma'^{n/2}$, for some $\epsilon_F > 0$, for all n, k. Hence, $P_k = \sum_{j=0}^{\infty} (F_k^T)^j (F_k)^j \le \lambda_m I$, where $\lambda_m := \frac{\epsilon_F^2}{1-\sigma'}$.

In what follows, we will first show that W_k has a bounded growth rate, and then that W_k has a certain "contraction" property, viz. $W_{k+T} < W_k$ for a certain T whenever W_k is large enough. These will then be used to prove the boundedness of W_k, and hence of all signals in the closed-loop system, thus establishing "robust boundedness" of the adaptive system.

Lemma (Bounded Growth Rate of W). *There exist constants K_w and k_w such that,*

$$W_k \le K_w W_{k-1} + k_w \text{ for all } k.$$

Proof First,

$$W_k \le \frac{k_F \epsilon_F \|x_k\|^2}{1-\sigma} + z_k \le \frac{k_F \epsilon_F}{1-\sigma}(y_k^2 + \|\phi_{k-1}\|^2 + \sum_{j=p+1}^{p'+\max\{p,d\}-1} y_{k-j}^2 + \sum_{j=\ell+d}^{\ell+p'+d-1} u_{k-j}^2) + z_k.$$

Hence, for some K_{wz} and k_{wz},

$$W_k \le K_{wz} z_k + k_{wz} \le K_{wz}(K_z z_{k-1} + k_z) + k_{wz} \le K_{wz} K_z W_{k-1} + K_{wz} k_z + k_{wz}. \qquad \Box$$

The Key Lemma. *For every constant L large enough, whenever there is an interval $[a, b]$ with $W_k \ge 2K_{wz}L$ for all $k \in [a, b]$, the following properties hold.*

i)

$$z_k \le W_k \le 2K_{wz} z_k \ \forall k \in [a, b].$$

ii)

$$W_k \le K_{ww} W_{k-1}, \forall k \in [a+1, b+1], \ \textit{where}\ K_{ww} := K_w + \frac{k_w}{2K_{wz}L}.$$

iii) If $I_{k-1} = 0$ *for all* $k \in [a, b]$*, then*

$$W_b \le 2K_{wz} \left(g^{b-a} + \frac{2K_3}{(1-g)L} \right) W_a.$$

iv) Let $\overline{\gamma} \in (\max(1 - \frac{1}{\lambda_m}, g), 1)$*. Suppose either that* $I_j = 1$ *for all* $j \in [a, b]$*, or if* $I_j = 0$ *for any* $j \in [a, b]$*, then suppose there exists an* N *such that* $I_{j+n} = 1$ *for some* $n \in [0, N]$*. Then there exists* $0 < \lambda < 1$ *such that,*

$$W_b \le K \exp[-(b-a)\lambda] \left[1 + \frac{K}{L\overline{\gamma}^{b-a}} \right] W_a.$$

Proof

i) We already have $z_k \le W_k$, by the definition of W_k. To show that $W_k \le 2K_{wz} z_k$ for L large enough, we note that because $W_k \le K_{wz} z_k + k_{wz}$, and $W_k \ge 2K_{wz} L$,

$$z_k \ge \frac{2K_{wz}L - k_{wz}}{K_{wz}} = 2L - \frac{k_{wz}}{K_{wz}}.$$

Hence if we choose L large enough that $k_{wz} \le K_{wz}[2L - \frac{k_{wz}}{K_{wz}}]$, then $k_{wz} \le K_{wz} z_k$, thus yielding $W_k \le 2K_{wz} z_k$.

ii) This follows easily from the lemma above.

iii) Since $I_{k-1} = 0$, we have $z_k = g z_{k-1} + 2K_3$ for $a \le k \le b$. Hence,

$$\begin{aligned} z_b &= g^{b-a} z_a + 2K_3 \sum_{j=a+1}^{b} g^{b-j} \le g^{b-a} W_a + \frac{2K_3}{(1-g)} \\ &\le [g^{b-a} + \frac{2K_3}{L(1-g)}] W_a, \end{aligned} \tag{38}$$

since $2K_{wz} \ge 1$ implies $W_a \ge 2K_{wz} L \ge L$. Hence $W_b \le 2K_{wz} z_b \le 2K_{wz}[g^{b-a} + \frac{2K_3}{L(1-g)}] W_a$.

iv) First, let us consider the case $d = 1$ for simplicity. We note that

$$\begin{aligned} W_k &= k_F x_k^T P_k x_k + z_k \\ &\le k_F (F_k x_{k-1} + b_{e,k} e_k + l_k)^T P_k (F_k x_{k-1} + b_{e,k} e_k + l_k) + g z_{k-1} \\ &\quad + K_y e_k^2 + K_u u_{k-1}^2 + 2K_3 \ (\text{with } l_k := b_{v,k} v_k + b_{r,k} r'_{k-1}) \\ &\le k_F x_{k-1}^T (P_k - I) x_{k-1} + 2k_F x_{k-1}^T F_k^T P_k b_{e,k} e_k + 2k_F x_{k-1}^T F_k^T P_k l_k \\ &\quad + 2k_F b_{e,k}^T P_k l_k e_k + k_F b_{e,k}^T P_k b_{e,k} e_k^2 + k_F l_k^T P_k l_k \\ &\quad + g z_{k-1} + K_y e_k^2 + K_u u_{k-1}^2 + 2K_3. \end{aligned} \tag{39}$$

Now defining $\gamma_1 := \sup_k \|F_k^T P_k b_{e,k}\|$, $\gamma_2 := \sup_k \|F_k^T P_k\|$, $\gamma_3 := \sup_k \|P_k b_{e,k}\|$, and since $P_k \leq \lambda_m I$, we have the following inequalities,

$$\left|2k_F x_{k-1}^T F_k^T P_k b_{e,k} e_k\right| \leq 2\gamma_1 k_F \|x_{k-1}\| |e_k| \leq \gamma_1 k_F \left[\epsilon_5 \|x_{k-1}\|^2 + \frac{1}{\epsilon_5} e_k^2\right],$$

$$\left|2k_F x_{k-1}^T F_k^T P_k l_k\right| \leq \gamma_2 k_F \left(\epsilon_6 \|x_{k-1}\|^2 + \frac{1}{\epsilon_6}\|l_k\|^2\right),$$

$$\left|2k_F b_{e,k}^T P_k l_k e_k\right| \leq \gamma_3 k_F \left(\|l_k\|^2 + e_k^2\right),$$

and

$$u_{k-1}^2 \leq \frac{1}{b_{\min}^2}\left(K + K(K_\theta + K_{\tilde{\theta}})^2 \|x_{k-1}\|^2\right) \text{ (from (26))}. \tag{40}$$

Hence,

$$\begin{aligned} W_k \leq\ & k_F x_{k-1}^T P_{k-1} x_{k-1} + g z_{k-1} + k_F[-1 + \epsilon_5\gamma_1 + \epsilon_6\gamma_2 + \frac{K_u K(K_\theta + K_{\tilde{\theta}})^2}{b_{\min}^2 k_F}]\|x_{k-1}\|^2 \\ & + k_F x_{k-1}^T (P_k - P_{k-1}) x_{k-1} + [\frac{\gamma_1 k_F}{\epsilon_5} + \gamma_3 k_F + \left(1 + \frac{1}{b_{\min}^2}\right) k_F \lambda_m + K_y] e_k^2 \\ & + \left[\frac{\gamma_2 k_F}{\epsilon_6} + \gamma_3 k_F + k_F \lambda_m\right] \|l_k\|^2 + 2K_3 + \frac{K K_u}{b_{\min}^2}. \end{aligned}$$

Now, choose $\overline{\gamma} \in (\max(1 - \frac{1}{\lambda_m}, g), 1)$, k_F large enough, and $\epsilon_5, \epsilon_6 > 0$ small enough so that,

$$-1 + \epsilon_5\gamma_1 + \epsilon_6\gamma_2 + \frac{K_u K(K_\theta + K_{\tilde{\theta}})^2}{b_{\min}^2 k_F} \leq -(1 - \overline{\gamma})\lambda_m, \tag{41}$$

and define $K_{ee} := \frac{\gamma_1 k_F}{\epsilon_5} + \gamma_3 k_F + \left(1 + \frac{1}{b_{\min}^2}\right) k_F \lambda_m + K_y$, $K_l := \frac{\gamma_2 k_F}{\epsilon_6} + \gamma_3 k_F + k_F \lambda_m$ and $\overline{K}_3 := K_3 + \frac{K K_u}{2b_{\min}^2}$. This gives,

$$\begin{aligned} W_k \leq\ & k_F x_{k-1}^T P_{k-1} x_{k-1} + g z_{k-1} - k_F(1 - \overline{\gamma})\lambda_m \|x_{k-1}\|^2 \\ & + k_F x_{k-1}^T (P_k - P_{k-1}) x_{k-1} + K_{ee} e_k^2 + K_l \|l_k\|^2 + 2\overline{K}_3. \end{aligned}$$

However,

$$\begin{aligned} k_F x_{k-1}^T P_{k-1} x_{k-1} + g z_{k-1} &= \overline{\gamma} W_{k-1} + k_F(1 - \overline{\gamma}) x_{k-1}^T P_{k-1} x_{k-1} + (g - \overline{\gamma}) z_{k-1} \\ &\leq \overline{\gamma} W_{k-1} + k_F (1 - \overline{\gamma}) \lambda_m \|x_{k-1}\|^2, \end{aligned}$$

and so,

$$W_k \leq \overline{\gamma} W_{k-1} + k_F x_{k-1}^T (P_{k-1} - P_{k-1}) x_{k-1} + K_{ee} e_k^2 + K_l \|\ell_k\|^2 + 2\overline{K}_3.$$

Now focusing on $x_{k-1}^T (P_k - P_{k-1}) x_{k-1}$, we have

$$x_{k-1}^T (P_k - P_{k-1}) x_{k-1} = x_{k-1}^T [\sum_{i=0}^{\infty} (F_k^T)^i F_k^i - \sum_{i=0}^{\infty} (F_{k-1}^T)^i F_{k-1}^i] x_{k-1}$$

$$
\begin{aligned}
&= \sum_{i=0}^{\infty} \left(\|F_k^i x_{k-1}\|^2 - \|F_{k-1}^i x_{k-1}\|^2\right) \\
&= \sum_{i=0}^{\infty} (\|F_k^i x_{k-1}\| + \|F_{k-1}^i x_{k-1}\|)(\|F_k^i x_{k-1}\| - \|F_{k-1}^i x_{k-1}\|) \\
&\leq \sum_{i=0}^{\infty} 2\epsilon_F \sigma^{i/2} \|x_{k-1}\| (\|F_k^i x_{k-1}\| - \|F_{k-1}^i x_{k-1}\|)
\end{aligned}
$$

Now note that,

$$
\begin{aligned}
\|F_k^i x_{k-1}\| &- \|F_{k-1}^i x_{k-1}\| \leq \|(F_k^i - F_{k-1}^i) x_{k-1}\| \leq \|F_k^i - F_{k-1}^i\| \|x_{k-1}\| \\
&= \|F_k^i - F_k^{i-1} F_{k-1} + F_k^{i-1} F_{k-1} - F_k^{i-2} F_{k-1}^2 \\
&\quad + F_k^{i-2} F_{k-1}^2 \ldots - F_{k-1}^i\| \|x_{k-1}\| \\
&\leq \sum_{j=0}^{i-1} \|F_k^{i-j} F_{k-1}^j - F_k^{i-1-j} F_{k-1}^{j+1}\| \|x_{k-1}\| \\
&\leq \sum_{j=0}^{i-1} \|F_k^{i-1-j}\| \|F_{k-1}^j\| \|F_k - F_{k-1}\| \|x_{k-1}\| \\
&\leq \epsilon_F^2 \sum_{j=0}^{i-1} \sigma^{(i-1-j)/2} \sigma^{j/2} K \|\delta_k\| \|x_{k-1}\| \\
&= i\sigma^{(i-1)/2} \epsilon_F^2 K \|\delta_k\| \|x_{k-1}\|.
\end{aligned}
$$

Hence

$$
\begin{aligned}
x_{k-1}^T (P_k - P_{k-1}) x_{k-1} &\leq K\epsilon_F^3 \sum_{i=0}^{\infty} 2i\sigma^{(2i-1)/2} \|\delta_k\| \|x_{k-1}\|^2 \leq K\epsilon_F^3 \|\delta_k\| \|x_{k-1}\|^2 \\
&\leq K\epsilon_F^3 \|\delta_k\| z_{k-1}.
\end{aligned}
$$

Thus we have,

$$
\begin{aligned}
W_k &\leq \overline{\gamma} W_{k-1} + K k_F \epsilon_F^3 \|\delta_k\| z_{k-1} + K_{ee} e_k^2 + K_l \|l_k\|^2 + 2\overline{K}_3 \\
&\leq [\overline{\gamma} + K k_F \epsilon_F^3 \|\delta_k\| + K_{ee} \frac{e_k^2}{z_{k-1}} + K_l \frac{\|l_k\|^2}{z_{k-1}}] W_{k-1} + 2\overline{K}_3,
\end{aligned}
$$

i.e., $W_k \leq g_k W_{k-1} + 2\overline{K}_3$ where g_k is the term in the square brackets above. From this recursive bound, we obtain,

$$
W_b \leq (\prod_{j=a+1}^{b} g_j) W_a + 2\overline{K}_3 \sum_{t=a+1}^{b} (\prod_{j=t+1}^{b} g_j),
$$

and so,

$$
\frac{W_b}{W_a} \leq \left(\prod_{j=a+1}^{b} g_j\right) \left[1 + \frac{2\overline{K}_3}{W_a} \sum_{t=a+1}^{b} \left(\prod_{j=a+1}^{t} g_j\right)^{-1}\right]
$$

$$
\begin{aligned}
&\leq \quad e^{\sum_{j=a+1}^{b} \ln g_j} \left[1 + \frac{\overline{K}_3}{K_{wz} L} \sum_{t=a+1}^{b} \left(\frac{1}{\overline{\gamma}} \right)^{t-a} \right] \\
&\leq \quad e^{\sum_{j=a+1}^{b} \ln g_j} \left[1 + \frac{\overline{K}_3}{K_{wz} L} \frac{1}{\overline{\gamma}^{b-a}(1-\overline{\gamma})} \right]. \qquad (42)
\end{aligned}
$$

Now, since $\log x \leq x - 1$, $\forall x > 0$,

$$
\begin{aligned}
\sum_{j=a+1}^{b} \ln g_j \quad &\leq \quad \sum_{j=a+1}^{b} g_j - (b-a) \\
&\leq \quad (\overline{\gamma} - 1)(b-a) + K k_F \epsilon_F^3 \sum_{j=a+1}^{b} \|\delta_j\| \\
&\quad + K_{ee} \sum_{j=a+1}^{b} \frac{e_j^2}{z_{j-1}} + K_l \sum_{j=a+1}^{b} \frac{\|l_j\|^2}{z_{j-1}}.
\end{aligned}
$$

Using vii) of Lemma 3 and the fact that $z_{j-i} \geq L$, we get $z_{j-1} \geq K_\rho \rho_{j-1}$, $\forall j \in [a+1, b]$. Now noting that $\sum_{j=a+1}^{b} \|\delta_j\| \leq K_\delta + k_\delta(b-a)$, and

$$
\begin{aligned}
\frac{\|l_j\|^2}{z_{j-1}} \quad &= \quad \frac{\|b_{v,j} v_j + b_{r,j} r'_{j-1}\|^2}{z_{j-1}} \leq 2 \left[\frac{v_j^2}{b_{\min}^2 z_{j-1}} + \frac{\|b_{r,j}\|^2 r'^2_{j-1}}{z_{j-1}} \right] \\
&\leq \quad 2 \frac{K_{vz}}{b_{\min}^2} + \frac{2k_{vz}}{b_{\min}^2 L} + \frac{2K(1 + K_{\hat{\theta}}^2)}{L},
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\sum_{j=a+1}^{b} \ln g_j \quad &\leq \quad -(1 - \overline{\gamma} - K k_F \epsilon_F^3 k_\delta)(b-a) + K k_F \epsilon_F^3 K_\delta + \frac{K_{ee}}{K_\rho} \sum_{j=a+1}^{b} \frac{e_j^2}{\rho_{j-1}} \\
&\quad + K_l \left[\frac{2K_{vz}}{b_{\min}^2} + \frac{2k_{vz}}{b_{\min}^2 L} + \frac{2K(1 + K_{\hat{\theta}}^2)}{L} \right] (b-a).
\end{aligned}
$$

From (17), using Lemma 4(ii), we get

$$
\begin{aligned}
\sum_{j=a+1}^{b} \frac{e_j^2}{\rho_{j-1}} \quad &\leq \quad K_{ev} \sum_{j=a+1}^{b} \frac{v_j^2}{\rho_{j-1}} + k_e(b-a) + K_e \\
&\leq \quad \frac{K_{ev}}{\bar{k}(N)} (b-a) [K_{vz} + \frac{k_{vz}}{L}] + k_e(b-a) + K_e.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\sum_{j=a+1}^{b} \ln g_j \quad &\leq \quad -(1 - \overline{\gamma} - K k_F \epsilon_F^3 k_\delta)(b-a) + K k_F \epsilon_F^3 K_\delta \\
&\quad + \frac{K_{ee}}{K_\rho} \frac{K_{ev}}{\bar{k}(N)} \left[K_{vz} + \frac{k_{vz}}{L} \right] (b-a) + \frac{K_{ee}}{K_\rho} k_e (b-a)
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{K_{ee}K_e}{K_\rho} + K_l\left[\frac{2K_{vz}}{b_{\min}^2} + \frac{2k_{vz}}{b_{\min}^2 L} + \frac{2K(1+K_{\hat{\theta}}^2)}{L}\right](b-a) \\
= \quad & -(b-a)\left(1-\overline{\gamma} - Kk_F\epsilon_F^3 k_\delta - \frac{K_{ee}}{K_\rho}\frac{K_{ev}}{\bar{k}(N)}\left[K_{vz} + \frac{k_{vz}}{L}\right]\right. \\
& \left. -\frac{K_{ee}k_e}{K_\rho} - K_l\left[\frac{2K_{vz}}{b_{\min}^2} + \frac{2k_{vz}}{b_{\min}^2 L} + \frac{2K(1+K_{\hat{\theta}}^2)}{L}\right]\right) \\
& +Kk_F\epsilon_F^3 K_\delta + \frac{K_{ee}K_e}{K_\rho} \\
= \quad & -(b-a)\left[1-\overline{\gamma} - Kk_F\epsilon_F^3 k_\delta - K_{vz}\left(\frac{K_{ee}}{K_\rho}\frac{K_{ev}}{\bar{k}(N)} + \frac{2K_l}{b_{\min}^2}\right)\right. \\
& \left. -\frac{1}{L}\left(\frac{K_{ee}}{K_\rho}\frac{K_{ev}}{\bar{k}(N)}k_{vz} + \frac{2k_{vz}K_l}{b_{\min}^2} + \frac{2K(1+K_{\hat{\theta}}^2)K_l}{L}\right) - \frac{K_{ee}k_e}{K_\rho}\right] \\
& +Kk_F\epsilon_F^3 K_\delta + \frac{K_{ee}K_e}{K_\rho}.
\end{aligned}
$$

By choosing K_v, k_δ small enough, and L large enough, we get

$$
\sum_{j=a+1}^{b} \ln g_j \leq -(b-a)\lambda + \frac{K_e K_{ee}}{K_\rho} + Kk_F\epsilon_F^3 K_\delta, \tag{43}
$$

for some $0 < \lambda < 1$.

We thus obtain,

$$
\frac{W_b}{W_a} \leq e^{-(b-a)\lambda}\left[1 + \frac{\overline{K}_3}{K_{wz}L(1-\overline{\gamma})\overline{\gamma}^{b-a}}\right] e^{\left(\frac{K_e K_{ee}}{K_\rho} + Kk_F\epsilon_F^3 K_\delta\right)}.
$$

Now we turn to the case $d > 1$. The essential difference between the cases $d = 1$ and $d > 1$ is that we replace $b_{e,k}e_k$ by $b_1 l_{1,k}$, and replace l_k by $b_{1,k-d}$ in (39). Hence we need to bound $l_{1,k}^2$, and $l_{2,k}^2$ in terms of $e_{(\cdot)}^2$, $v_{(\cdot)}^2$, and $\|\delta_{(\cdot)}\|$. This is done as in the proof of Lemma 3, except that $A^*(B_{k-1} - B_{k-d})u_{k-d}$, which is part of $\Delta_{5,k}u_{k-d}$ has to be handled differently. It gives rise to terms involving $\|\delta_j\|$.

Let us consider how one overbounds $l_{1,k}^2$. As in the proof of Lemma 3(ii), we obtain,

$$
\begin{aligned}
(\Delta_{1,k}u_{k-d})^2 &\leq K\sum_{j=1}^{d-1} e_{k-j}^2 + K, \\
(\Delta_{2,k}u_{k-d})^2 &\leq K\sum_{j=1}^{d-1} e_{k-j}^2 + K, \\
(\Delta_{3,k}y_k)^2 &\leq K\sum_{j=1}^{d-1} e_{k-j}^2 + K, \\
\text{and } (\Delta_{4,k}y_{k-d})^2 &\leq K\sum_{j=1}^{d-1} e_{k-j}^2 + K.
\end{aligned}
$$

Combining these yields $l_{1,k}^2 \leq K \sum_{j=1}^{d-1} e_{k-j}^2 + K$.

Next, consider $l_{2,k}^2$. First, note that we have the following inequality, whose counterpart in the $d=1$ case is (40).

$$u_{k-d}^2 \leq K\|x_{k-1}\|^2 + K. \tag{44}$$

This is obtained by applying the control law (9), using the boundedness of the parameter estimates, and the boundedness from below, by $b_{\min} > 0$, of $\hat{b}_{1,k-d}$.

For the term $A^*(B_{k-1} - B_{k-d})u_{k-d}$, we have

$$\begin{aligned}
(A^*(B_{k-1} - B_{k-d})u_{k-d})^2 &= ((B_{k-1} - B_{k-d})A^* u_{k-d})^2 \\
&= (\sum_{j=1}^{d-1}(B_{k-j} - B_{k-j-1})A^* u_{k-d})^2 \\
&= (\sum_{j=1}^{d-1}\sum_{i=0}^{\ell-1}\sum_{i'=1}^{p'}(b_{i+1,k-j} - b_{i+1,k-j-1})a_{i'}^* u_{k-d-i-i'})^2 \\
&\leq K(\sum_{j=1}^{d-1}\|\theta_{k-j} - \theta_{k-j-1}\| \sum_{i=0}^{p'+\ell-1}|u_{k-d-i}|)^2 \\
&\leq K\sum_{j=1}^{d-1}\|\theta_{k-j} - \theta_{k-j-1}\|^2(u_{k-d}^2 + \ldots + u_{k-d-p'-\ell+1}^2) \\
&\leq (K\|x_{k-1}\|^2 + K)\left(\sum_{j=1}^{d-1}\|\theta_{k-j} - \theta_{k-j-1}\|^2\right) \\
&\leq (Kz_{k-1} + K)\left(\sum_{j=1}^{d-1}\|\theta_{k-j} - \theta_{k-j-1}\|^2\right),
\end{aligned}$$

where the last two inequalities follow from (44), and the definition of $z(\cdot)$. The remaining terms can be handled similarly. □

The Contraction Lemma

Consider $0 < \gamma^* < 1$. Then there exist N, L large enough, and $K_{v\max}, k_{\delta\max} > 0$, so that if $K_v \in [0, K_{v\max}]$ and $k_\delta \in [0, k_{\delta\max}]$, and

$$W_k \geq 2K_{wz}L, \text{ for all } k \in [l-1, l+2N],$$

then

$$W_{l+2N} \leq \gamma^* W_{l-1}. \tag{45}$$

Proof There are four cases to consider.

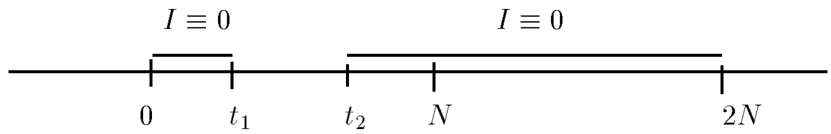

Figure 1: Illustration of Case 2

Case 1: Suppose $I_{t-1} = 0, \forall t \in [l, l+2N]$. From part (iii) of the Key Lemma,

$$W_{l+2N} \leq 2K_{wz}\left(g^{2N} + \frac{2\overline{K}_3}{(1-g)L}\right) W_l.$$

Hence, for N, L large enough, and $K_{v\,\max}$ and $k_{\delta\,\max}$ correspondingly small, we have (45).

Case 2: Suppose $0 \leq t_1 \leq t_2 \leq N$ where $t_1 = \min\{t \in [0, 2N] : I_{t+l-1} = 1\}$ and $t_2 = \max\{t \in [t_1, 2N] : I_{t+l-1} = 1\}$.

Note that this implies that $I_{k-1} = 0, \forall k \in [l, l+t_1-1]$, and $I_{k-1} = 0, \forall k \in [l+t_2+1, l+2N]$. First suppose $t_2 > 0$. Using parts (ii), (iii) and (iv) of the Key Lemma, we have

$$\begin{aligned}
W_{l+2N} &\leq 2K_{wz}\left(g^{2N-t_2-1} + \frac{2\overline{K}_3}{(1-g)L}\right) W_{l+t_2+1} \\
&\leq 2K_{wz}K_{ww}^2\left(g^{2N-t_2-1} + \frac{2\overline{K}_3}{(1-g)L}\right) W_{l+t_2-1} \\
&\leq 2K_{wz}K_{ww}^2\left(g^{N-1} + \frac{2\overline{K}_3}{(1-g)L}\right) K\exp[-\lambda t_2]\left(1 + \frac{K}{L\overline{\gamma}^{t_2}}\right) W_{l-1} \\
&\leq 2K_{wz}K_{ww}^2 K\left(g^{N-1} + \frac{2\overline{K}_3}{(1-g)L}\right)\left(1 + \frac{K}{L\overline{\gamma}^{N}}\right) W_{l-1},
\end{aligned}$$

yielding (45) when N, L are large enough, and $K_{v\,\max}$ and $k_{\delta\,\max}$ appropriately small. If $t_2 = 0$, then by parts (ii) and (iii) of the Key Lemma,

$$\begin{aligned}
W_{l+2N} &\leq 2K_{wz}\left(g^{2N-1} + \frac{2\overline{K}_3}{(1-g)L}\right) W_{l+1} \\
&\leq 2K_{wz}K_{ww}^2\left(g^{2N-1} + \frac{2\overline{K}_3}{(1-g)L}\right) W_{l-1},
\end{aligned}$$

hence giving (45).

Case 3: Suppose $N < t_1 \leq t_2 \leq 2N$. First suppose $t_2 < 2N$.

Using parts (ii), (iii) and (iv) of the Key Lemma, we have

$$\begin{aligned}
W_{l+2N} &\leq 2K_{wz}\left(g^{2N-t_2-1} + \frac{2\overline{K}_3}{(1-g)L}\right) W_{l+t_2+1}, \\
W_{l+t_2+1} &\leq K_{ww}^2 W_{l+t_2-1}, \\
W_{l+t_2-1} &\leq K\exp\left[-\lambda(t_2-1-N)\right]\left[1 + \frac{K}{L\overline{\gamma}^{t_2-1-N}}\right] W_{l+N},
\end{aligned}$$

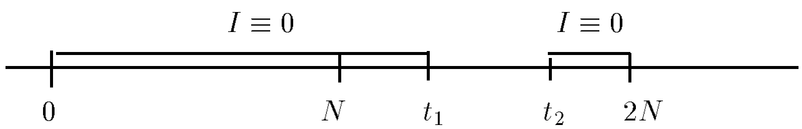

Figure 2: Illustration of Case 3

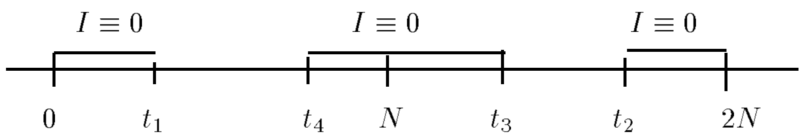

Figure 3: Illustration of Case 4

and using parts (ii) and (iii) of the Key Lemma gives

$$W_{l+N} \leq 2K_{wz}K_{ww}\left(g^N + \frac{2\overline{K}_3}{(1-g)L}\right)W_{l-1}.$$

This yields,

$$W_{l+2N} \quad \leq \quad 4KK_{wz}^2K_{ww}^3\left(g^N + \frac{2\overline{K}_3}{(1-g)L}\right)\left(1 + \frac{2\overline{K}_3}{(1-g)L}\right)\left(1 + \frac{K}{L\overline{\gamma}^{N-1}}\right)W_{l-1},$$

and hence (45). If $t_2 = 2N$, the first two inequalities get replaced by

$$W_{l+2N} \leq K_{ww}W_{l+2N-1} = K_{ww}W_{l+t_2-1}.$$

This gives

$$W_{l+2N} \quad \leq \quad 2KK_{wz}K_{ww}^2\left(g^N + \frac{2\overline{K}_3}{(1-g)L}\right)\left(1 + \frac{K}{L\overline{\gamma}^{N-1}}\right)W_{l-1},$$

hence giving (45).

Case 4: Suppose $0 \leq t_1 \leq N < t_2 \leq 2N$. Define $t_3 := \min\{t \in [N, t_2] : I_{t+l-1} = 1\}$, and $t_4 := \max\{t \in [t_1, N] : I_{t+l-1} = 1\}$. Note that this implies that $I_{k-1} = 0, \forall k \in [l + t_4 + 1, l + t_3 - 1]$.

Case 4a: Suppose $t_3 - t_4 < N$. First suppose $t_2 < 2N$. Using parts (ii), (iii) and (iv) of the Key Lemma, we have

$$W_{l+2N} \leq 2K_{wz}\left(g^{2N-t_2-1} + \frac{2\overline{K}_3}{(1-g)L}\right)W_{l+t_2+1},$$

$$W_{l+t_2+1} \leq K_{ww}^2 W_{l+t_2-1},$$

and

$$W_{l+t_2-1} \leq K \exp\left[-\lambda t_2\right] \left(1 + \frac{K}{L\overline{\gamma}^{t_2}}\right) W_{l-1},$$

which gives

$$W_{l+2N} \quad \leq \quad 2KK_{wz}K_{ww}^2 \left(1 + \frac{2\overline{K}_3}{(1-g)L}\right) \exp(-\lambda N) \left(1 + \frac{K}{L\overline{\gamma}^{2N}}\right) W_{l-1},$$

and hence (45). If $t_2 = 2N$, the first two inequalities get replaced by

$$W_{l+2N} \leq K_{ww} W_{l+2N-1} = K_{ww} W_{l+t_2-1}.$$

This gives

$$W_{l+2N} \leq KK_{ww} \exp(-2\lambda N) \left(1 + \frac{K}{L\overline{\gamma}^{2N}}\right) W_{l-1},$$

and hence (45).

Case 4b: Suppose $t_3 - t_4 \geq N$. Again, first suppose that $t_2 < 2N$. Using parts (ii), (iii) and (iv) of the Key Lemma, we have

$$\begin{aligned}
W_{l+2N} &\leq 2K_{wz}\left(g^{2N-t_2-1} + \frac{2\overline{K}_3}{(1-g)L}\right) W_{l+t_2+1}, \\
W_{l+t_2+1} &\leq K_{ww}^2 W_{l+t_2-1}, \\
W_{l+t_2-1} &\leq K \exp(-\lambda(t_2 - t_3)) \left(1 + \frac{K}{L\overline{\gamma}^{t_2-t_3}}\right) W_{l+t_3-1}, \\
W_{l+t_3-1} &\leq 2K_{wz}\left(g^{t_3-t_4-2} + \frac{2\overline{K}_3}{(1-g)L}\right) W_{l+t_4+1}, \\
W_{l+t_4+1} &\leq K_{ww}^2 W_{l+t_4-1},
\end{aligned}$$

and

$$W_{l+t_4-1} \leq K \exp(-\lambda t_4) \left(1 + \frac{K}{L\overline{\gamma}^{t_4}}\right) W_{l-1},$$

which gives

$$W_{l+2N} \quad \leq \quad 4K^2K_{wz}^2K_{ww}^4 \left(g^{N-2} + \frac{2\overline{K}_3}{(1-g)L}\right) \left(1 + \frac{K}{L\overline{\gamma}^{N}}\right)^2 \left(1 + \frac{2\overline{K}_3}{(1-g)L}\right) W_{l-1},$$

and hence (45). If $t_2 = 2N$, the first two inequalities get replaced by

$$W_{l+2N} \leq K_{ww} W_{l+2N-1} = K_{ww} W_{l+t_2-1}.$$

This gives

$$W_{l+2N} \leq 2K^2K_{wz}K_{ww}^3 \left(g^{N-2} + \frac{2\overline{K}_3}{(1-g)L}\right) \left(1 + \frac{K}{L\overline{\gamma}^{N}}\right)^2 W_{l-1},$$

thus establishing (45) in all cases. □

Theorem 1 (Robust Boundedness Theorem). *Consider the adaptive control system with the plant satisfying assumptions (A1-A5). Then, all signals in the closed-loop adaptive system are bounded, whenever K_v and k_δ are small enough.*

Proof As a consequence of the Bounded Growth Rate Lemma and the Contraction Lemma, W is bounded. To see this, note first that the Contraction Lemma shows that if W stays above a certain value $(2K_{wz}L)$ continuously for a certain time interval ($2N+1$ samples), it must contract. That is, W cannot continue to grow for more than $2N+1$ consecutive samples at a time, once it has grown larger that $2K_{wz}L$. Since W has a bounded growth rate, it follows that it is bounded.

Since W bounds all other signals through z and x, we conclude that all closed-loop signals are bounded. □

Remark: These results extend easily to recursive least-squares based schemes which keep the condition number of the covariance matrix bounded.

8 Performance for a Nominal System

In this section, we consider the performance that can be achieved in the absence of unmodelled dynamics, and when the parameter variations go to zero asymptotically, specifically when $\|\delta_k\| \in \ell_1$. We show that the desired performance objective is met.

Theorem 2. *If $K_v = k_v = k_\delta = 0$, then*

$$\text{(i)} \quad \lim_{k\to\infty} e_k = 0$$

$$\text{(ii)} \quad \lim_{k\to\infty} (A^*(q^{-1})y_k - q^{-d}B^*(q^{-1})r_k) = 0.$$

Remark: Note that we do not need the nominal plant parameters to be time-invariant; we only require them to satisfy $\|\delta_k\| \in \ell_1$. That is,

$$\sum_{k=t+1}^{\infty} \|\theta_k - \theta_{k-1}\| \le K_\delta, \ \forall t.$$

Proof

(i) Recall from Lemma 1(iii) that

$$\sum_{k=t+1}^{t+T} \frac{e_k^2}{\rho_{k-1}} \le K_{ev} \sum_{k=t+1}^{t+T} \frac{v_k^2}{\rho_{k-1}} + k_e T + K_e.$$

Since $K_v = K_v = k_\delta = 0$, from (A4) and Lemma 1(iii) we have, $v_k \equiv 0$, and $k_e = 0$, which gives

$$\sum_{k=t+1}^{t+T} \frac{e_k^2}{\rho_{k-1}} \le K_e, \ \forall t, T.$$

Recalling that $\rho_{(\cdot)}$ is uniformly bounded by Theorem 1, and letting $\rho_{\max} := \max_{k \geq 0}\{\rho_k\}$, we get

$$\sum_{k=t+1}^{t+T} e_k^2 \leq K_e \rho_{\max}, \ \forall t, T.$$

Fixing t and letting T go to infinity gives $e \in \ell_2$ and hence $e_k \to 0$ as $k \to \infty$.

(ii) Recalling (10), the fact that $\|\widehat{\theta}_k - \widehat{\theta}_{k-1}\| \leq \|\widehat{\theta}'_k - \widehat{\theta}_{k-1}\|$ due to parameter projection, and using the Cauchy-Schwarz inequality, we get,

$$\|\widehat{\theta}_k - \widehat{\theta}_{k-j}\| \in \ell_2, \quad \text{for all finite } j.$$

Using this in (27), the closed-loop boundedness of all signals, and using (1)(a) yields the desired result. □

9 Robust Performance

We now show that the performance of the adaptive controller, as measured by the mean square output prediction error, is robust in that it is linear (hence also continuous) in the magnitude of the unmodelled dynamics, bounded disturbances, and average rate of parameter variations.

Theorem 3.

$$\limsup_{T \to \infty} \frac{1}{T} \sum_{k=t+1}^{t+T} e_k^2 \leq c_1 K_v + c_2 k_v + c_3 k_\delta,$$

where c_1, c_2, c_3 are generic constants which can only decrease (or remain constant) as K_v, k_v, k_δ decrease.

Proof Using (A4) in Lemma 1(iii) and recalling that $\{m_k\}$ and $\{\rho_k\}$ are uniformly bounded by Theorem 1, we get,

$$\begin{aligned} \sum_{k=t+1}^{t+T} \frac{e_k^2}{\rho_{k-1}} &\leq K_{ev} \sum_{k=t+1}^{t+T} \frac{K_v m_{k-1} + k_v}{\rho_{k-1}} + k_e T + K_e, \ \forall t, T \\ &\leq K_{ev}(K K_v + K k_v) T + k_e T + K_e, \ \forall T. \end{aligned}$$

This implies

$$\limsup_{T \to \infty} \frac{1}{T} \sum_{k=t+1}^{t+T} e_k^2 \leq K K_{ev}(K K_v + K k_v) + K k_e,$$

which yields the desired result after using Lemma 1(iii). □

10 Simulation Example

We now demonstrate the advantage of the suggested adaptive control algorithm through a simulation example. The system is modelled as

$$y_k = a_k y_{k-1} + b_k u_{k-1}$$

where a_k, and b_k are unknown, possibly time-varying parameters. The true unknown system is, however, given by the following.

$$y_k \quad = \quad 1.5\sin\left(\frac{\pi k}{1000}\right) y_{k-1} + \left[1 + 0.4\cos\left(\frac{\pi k}{1500}\right)\right] u_{k-1} + 0.2\sum_{j=0}^{k-2}(0.5)^j y_{k-2-j} + d_k,$$

where d_k denotes a discrete square wave disturbance of period 100 and amplitude 0.15. The adaptive control is designed to track the following model reference trajectory,

$$y_k^* = -0.5 y_{k-1}^* + r_k.$$

As Fig. 4 shows, using an unmodified LMS-type gradient estimator or an LMS-type gradient estimator with parameter projection, causes the output to blow up. Fig. 5 illustrates the undesirable behavior that results if we use a normalized gradient estimator, as used in the ideal case. Finally, Fig. 6 illustrates the results obtained if we use the adaptive control algorithm with parameter projection, as proposed in this paper. Fig. 6(b) also exhibits the nice parameter tracking that is achieved.

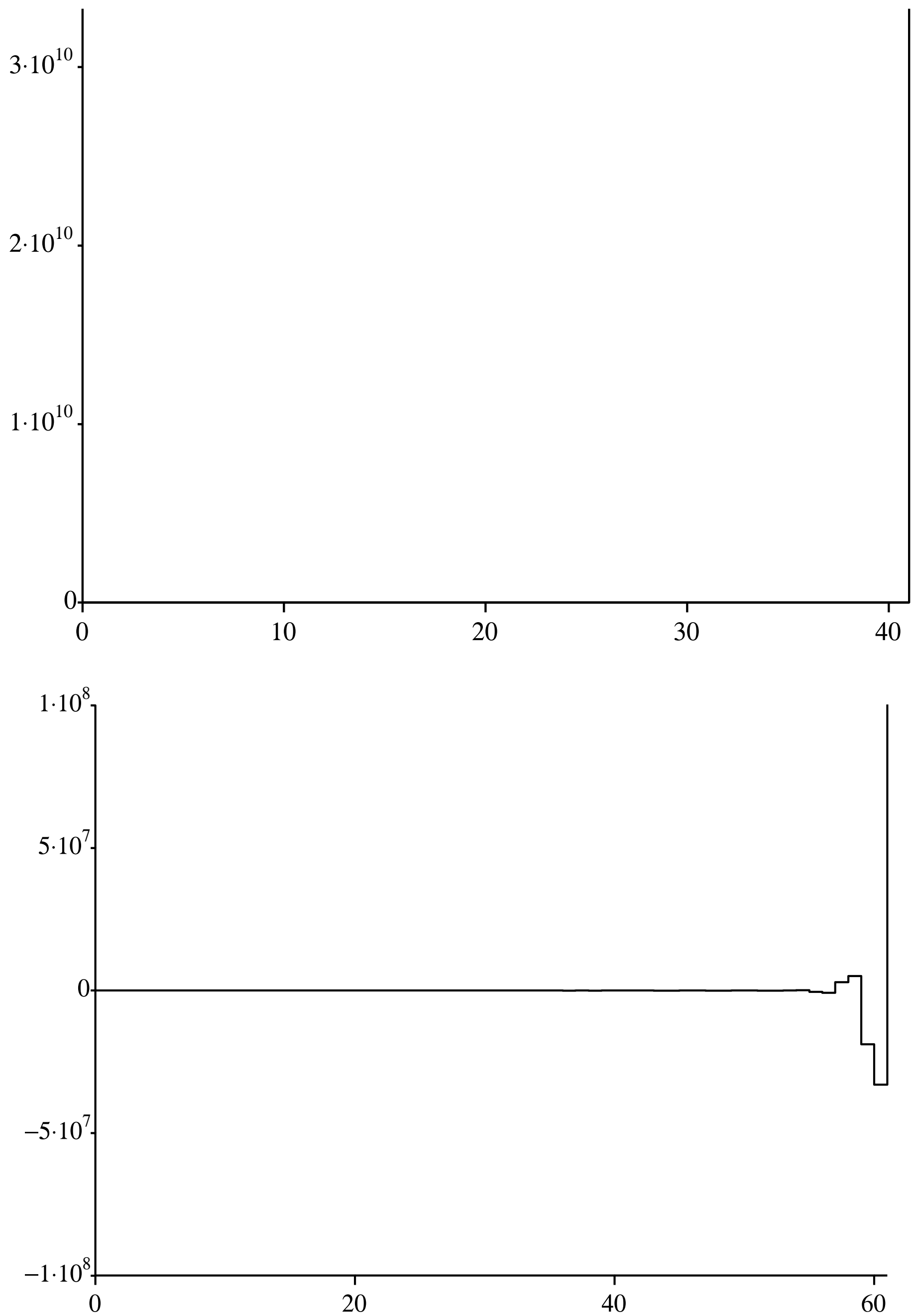

Figure 4: (a) Unmodified (LMS-type) gradient estimator: y blows up (b) Gradient estimator with parameter projection: y blows up

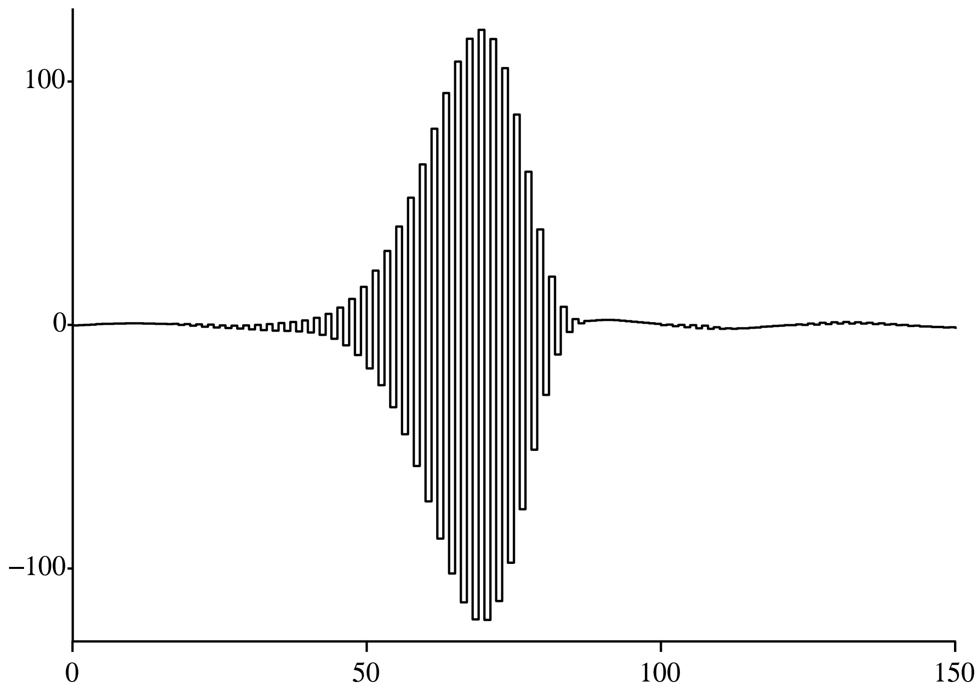

Figure 5: Normalized gradient estimator: y is not well-behaved

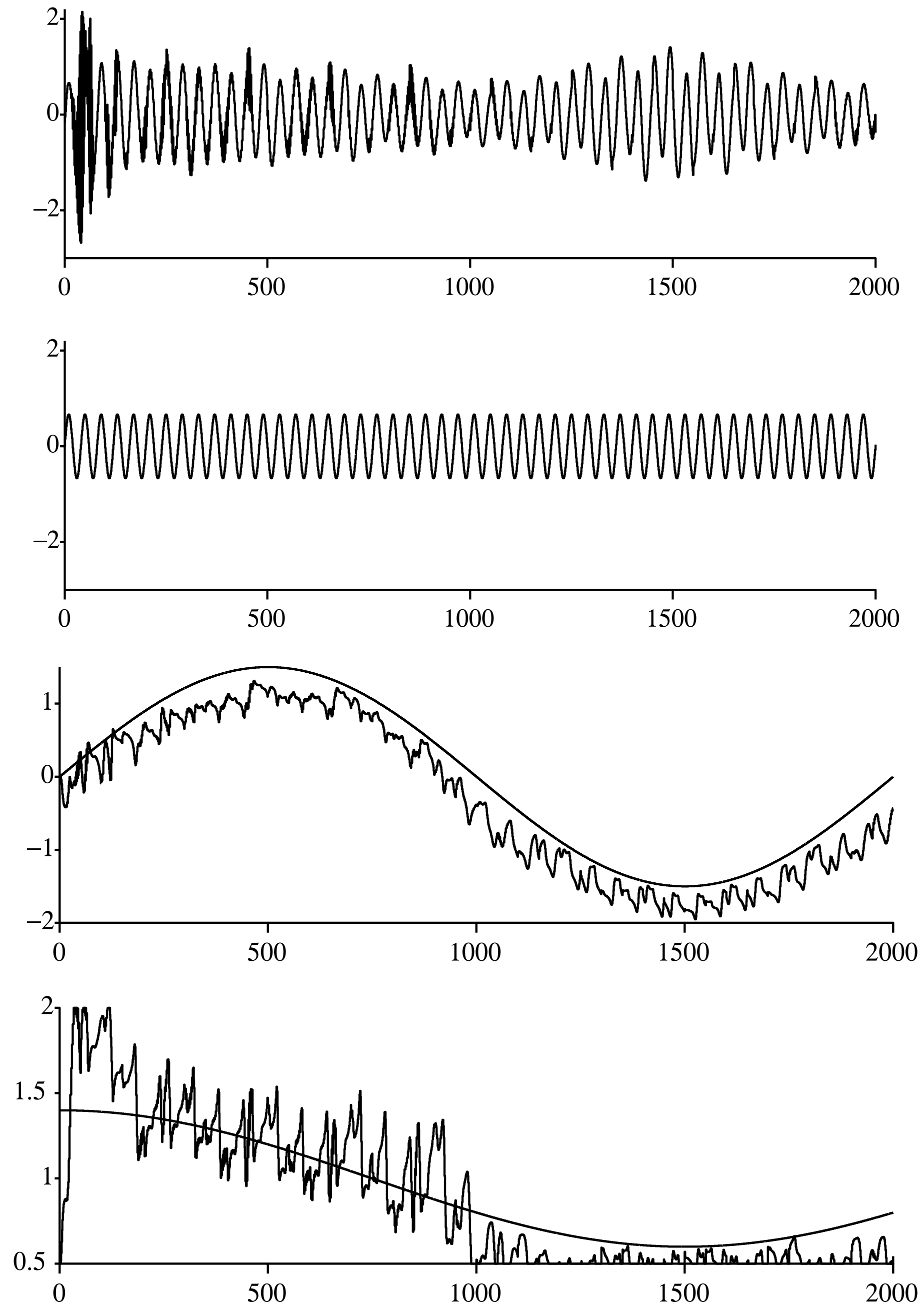

Figure 6: Proposed adaptive control algorithm: (a)(i) System output, y (ii) Model reference output, y^* (b)(i) True and estimated a_k (ii) True and estimated b_k

11 Concluding Remarks

We have presented an indirect adaptive pole-zero placement control law using a simple parameter estimator employing projection. We have shown that this is robust for plants which simultaneously feature unknown slow-in-the-mean time-variations of the nominal parameters, as well as small unmodeled dynamics, and bounded disturbances, without any restriction on the magnitude of the bound. The plant parameters may even make occasional jumps. No special normalization is used. Instead, the signals entering the parameter update law are normalized by the squared norm of an "extended" regressor, which requires neither any a priori system knowledge nor any additional computation.

It is straightforward to extend this analysis to recursive least-squares based update laws which monitor, and keep bounded, the condition number of the covariance matrix.

Several issues still need to be explored. A major restriction is that we require the frozen nominal plant to be minimum phase at every instant. Transient performance, and the precise sizes of unmodeled dynamics and parameter variations tolerated, are issues which require deeper study.

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