Critical Power for Asymptotic Connectivity in Wireless Networks

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ABSTRACT: In wireless data networks each transmitter's power needs to be high enough to reach the intended receivers, while generating minimum interference on other receivers sharing the same channel. In particular, if the nodes in the network are assumed to cooperate in routing each others' packets, as is the case in *ad hoc* wireless networks, each node should transmit with just enough power to guarantee connectivity in the network. Towards this end, we derive the *critical* power a node in the network needs to transmit in order to ensure that the network is connected with probability one as the number of nodes in the network goes to infinity. It is shown that if n nodes are placed in a disc of unit area in $\Re^2$ and each node transmits at a power level so as to cover an area of $\pi r^2 = (\log n + c(n))/n$, then the resulting network is asymptotically connected with probability one if and only if $c(n) \to +\infty$.

1 Introduction

Wireless communication systems consist of nodes which share a common communication medium: namely, radio. Signals intended for a receiver cause interference at other receiver nodes. This results in reduced signal to noise ratio at the latter receivers, and thus, in the lowering of their information-processing capacity. Hence, it becomes essential to control the transmitter power such that the information signals reach their intended receivers, while causing minimal interference for other receivers sharing the same channel. To achieve this objective, many iterative power control algorithms have been developed (Bambos, Chen and Pottie (1995), Ulukus and Yates (1996) and the references therein).

In this paper we look at the problem from a different perspective. We assume that nodes in the network cooperate in routing each others' data packets. Examples of such networks are mobile *ad hoc* networks (Gupta and

Kumar (1996) and Johnson and Maltz (1996)). They are networks formed by a group of mobile nodes which communicate with each other over a wireless channel and without any centralized control. In such networks, a critical requirement is that each node in the network has a *path* to every other node in the network, i.e., the network is connected. With this in mind, we consider the problem of determining the *critical* power at which each node needs to transmit so as to guarantee asymptotic connectivity of the network.

More precisely, we consider the following problem: Let $\mathcal{D}$ be a disc in $\Re^2$ having unit area. Let $\mathcal{G}(n, r(n))$ be the network (graph) formed when n nodes are placed uniformly and independently in $\mathcal{D}$, and two nodes i and j can communicate with each other if the distance between them is less than $r(n)$. That is, if x_k is the location of node k, nodes i and j can communicate if $\|x_i - x_j\| \leq r(n)$, where the norm used is the Euclidean norm (i.e., L^2-norm). The radius $r(n)$ is usually referred to as the *range* of a node in $\mathcal{G}(n, r(n))$. Then the problem is to determine $r(n)$ which guarantees that $\mathcal{G}(n, r(n))$ is asymptotically connected with probability one, i.e., the probability that $\mathcal{G}(n, r(n))$ is connected, denoted by $P_C(n, r(n))$, goes to one as $n \to \infty$. For this problem, we show that if $\pi r^2(n) = \frac{\log n + c(n)}{n}$, then $P_C(n, r(n)) \to 1$ if and only if $c(n) \to +\infty$.

A related problem that has been considered in the literature is connectivity in *Bernoulli* graphs: Let $\mathcal{B}(n, p(n))$ be a graph consisting of n nodes, in which edges are chosen independently and with probability $p(n)$. Then, it has been shown that if $p(n) = \frac{\log n + c(n)}{n}$, the probability that $\mathcal{B}(n, p(n))$ is connected goes to one if and only if $c(n) \to +\infty$ (Theorem VII.3 in Bollobás (1985)). Even though the asymptotic expression is the same, connectivity in $\mathcal{G}(n, r(n))$ is quite different from connectivity in $\mathcal{B}(n, p(n))$. The event that there are links between i and j, and between j and k, is not independent of the event that there is a link between i and k (as, fixing x_i, the former is true given the latter only if j lies in the intersection of two discs of radius $r(n)$ and centered at i and k, with $\|x_i - x_k\| \leq r(n)$. This has lower probability than the probability $(\pi r^2(n))^2$ of the event that there are links between i and j, and j and k). As it turns out, an entirely different proof technique was needed to prove asymptotic connectivity in $\mathcal{G}(n, r(n))$.

Another closely related problem considered in the literature is the *coverage* problem: Disks of radius a are placed in a unit-area disc $\mathcal{D} \in \Re^2$ at a Poisson intensity of λ, i.e., number of discs having their centers in a set $\mathcal{A} \subset \mathcal{D}$ of area $|\mathcal{A}|$ is Poisson distributed with mean $\lambda|\mathcal{A}|$. Let $\mathcal{V}(\lambda, a)$ denote the vacancy within $\mathcal{D}$, i.e., $\mathcal{V}(\lambda, a)$ is the region of $\mathcal{D}$ not covered by the disks. Then it has been shown in Hall (1988) (Theorem 3.11) that

$$\begin{aligned} \frac{1}{20}\min\left\{1, (1 + \pi a^2 \lambda^2) e^{-\pi a^2 \lambda}\right\} &< P(|\mathcal{V}(\lambda, a)| > 0) \\ &< \min\left\{1, 3(1 + \pi a^2 \lambda^2) e^{-\pi a^2 \lambda}\right\}. \quad (1.1) \end{aligned}$$

Note that (1.1) has more stringent asymptotics on $a(n)$ than our result. If $\lambda = n$ and $\pi a^2(n) = \frac{\log n + \log\log n + c(n)}{n}$, then $\lim_{n\to\infty} P(|\mathcal{V}(n, a(n))| > 0) = 0$ for $c(n) \to +\infty$, and $\lim_{n\to\infty} P(|\mathcal{V}(n, a(n))| > 0) \geq 1/20$ for $c(n) \to -\infty$. Also, note that coverage of $\mathcal{D}$ by discs of radius $a(n) = r(n)$ does not guarantee connectivity in $\mathcal{G}(n, r(n))$ (recall $r(n)$ is the range of nodes in $\mathcal{G}(n, r(n))$)). However, $a(n) = r(n)/2$ does; the corresponding lower bound on $r(n)$ is $\pi r^2(n) = 4\frac{\log n + \log\log n + c(n)}{n}$ for $c(n) \to +\infty$, which is much weaker than the one we obtain. Moreover, since $\mathcal{G}(n, r(n))$ can be connected without $\mathcal{D}$ being entirely covered by n discs of radius $r(n)$, this approach does not lead to any necessary conditions on $r(n)$ for asymptotic connectivity in $\mathcal{G}(n, r(n))$.

Yet another related problem considered is in *continuum percolation* theory (Kesten (1982), Mesteer and Roy (1996)): Nodes are assumed to be distributed with Poisson intensity λ in $\Re^2$, and two nodes are connected to each other if the distance between them is less than r. Then the problem considered is to find a critical value of r such that the origin is connected to an infinite-order component. Of course for this to make sense, the node distribution process is conditioned on the origin having a node. We will, in fact, make use of some results from percolation theory while deriving the sufficient condition on $r(n)$ for asymptotic connectivity in $\mathcal{G}(n, r(n))$ (cf. Section 3).

The rest of the paper is organized as follows. In Section 2 we derive the necessary condition on $r(n)$ for asymptotic connectivity of $\mathcal{G}(n, r(n))$. The sufficiency of this condition is proved in Section 3. We conclude in Section 4 with some comments on extensions of the problem considered.

2 Necessary Condition on $r(n)$ for Connectivity

In this section we derive necessary conditions on the radio range of a node in the network for asymptotic connectivity. In the following, to avoid technicalities which obscure the main ideas, we will neglect edge effects resulting due to a node being close to the boundary of $\mathcal{D}$. The complete proofs which take the edge effects into account are given in the Appendix.

We will frequently use the following bounds.

Lemma 2.1 *(i) For any $p \in [0, 1]$*

$$(1 - p) \leq e^{-p}. \tag{1.2}$$

(ii) For any given $\theta \geq 1$, there exists $p_0 \in [0, 1]$, such that

$$e^{-\theta p} \leq (1 - p), \quad \textit{for all } 0 \leq p \leq p_0. \tag{1.3}$$

If $\theta > 1$, then $p_0 > 0$.

Lemma 2.2 *If* $\pi r^2(n) = \frac{\log n + c}{n}$, *then, for any fixed* $\theta < 1$ *and for all sufficiently large* n

$$n(1 - \pi r^2(n))^{n-1} \geq \theta e^{-c}. \tag{1.4}$$

Proof: Taking the logarithm of the left hand side of (1.4), we get

$$\log(\text{L.H.S. of } (1.4)) = \log n + (n-1)\log\left(1 - \pi r^2(n)\right).$$

Using the power series expansion for $\log(1-x)$,

$$\begin{aligned}\log(\text{L.H.S. of } (1.4)) &= \log n - (n-1)\sum_{i=1}^{\infty}\frac{\left(\pi r^2(n)\right)^i}{i} \\ &= \log n - (n-1)\left(\sum_{i=1}^{2}\frac{(\log n + c)^i}{in^i} + \mathcal{E}(n)\right), \end{aligned} \tag{1.5}$$

where

$$\begin{aligned}\mathcal{E}(n) &= \sum_{i=3}^{\infty}\frac{(\log n + c)^i}{in^i} \\ &\leq \frac{1}{3}\int_2^{\infty}\left(\frac{\log n + c}{n}\right)^x dx \\ &= \frac{1}{3\log\left(\frac{\log n + c}{n}\right)}\left(\frac{\log n + c}{n}\right)^x\Bigg|_2^{\infty} \\ &\leq \frac{1}{3}\left(\frac{\log n + c}{n}\right)^2, \end{aligned} \tag{1.6}$$

for all large n. Substituting (1.6) in (1.5), we get

$$\begin{aligned}\log(\text{L.H.S. of } (1.4)) &\geq \log n - (n-1)\left(\frac{\log n + c}{n} + \frac{5(\log n + c)^2}{6n^2}\right) \\ &\geq -c - \frac{(\log n + c)^2 - (\log n + c)}{n} \\ &\geq -c - \epsilon, \end{aligned}$$

for all sufficiently large n. The result follows by taking the exponent of both sides and using $\theta = e^{-\epsilon}$. □

Now, let $P^{(k)}(n, r(n)), k = 1, 2, \ldots$ denote the probability that a graph $\mathcal{G}(n, r(n))$ has at least one order-k component. By an order-k component we mean a set of k nodes which form a connected set, but which are not connected with any other node. Also, let $P_d(n, r(n))$ denote the probability that $\mathcal{G}(n, r(n))$ is disconnected.

Theorem 2.1 *If* $\pi r^2(n) = \frac{\log n + c(n)}{n}$, *then*

$$\liminf_{n\to\infty} P_d(n, r(n)) \geq e^{-c}\left(1 - e^{-c}\right), \tag{1.7}$$

where $c = \lim_{n\to\infty} c(n)$.

Proof: We first study the case where $\pi r^2(n) = \frac{\log n + c}{n}$ for a fixed c. Consider $P^{(1)}(n, r(n))$, the probability that $\mathcal{G}(n, r(n))$ has *at least* one order-1 component. That is, $P^{(1)}(n, r(n))$ is the probability that $\mathcal{G}(n, r(n))$ has at least one node which does not include any other node in its range. Then

$$\begin{aligned} P^{(1)}(n, r(n)) &\geq \sum_{i=1}^{n} P(\{i \text{ is the only isolated node in } \mathcal{G}(n, r(n))\}) \\ &\geq \sum_{i=1}^{n} \Big(P(\{i \text{ is an isolated node in } \mathcal{G}(n, r(n))\}) \\ &\quad - \sum_{j\neq i} P(\{i \text{ and } j \text{ are isolated nodes in } \mathcal{G}(n, r(n))\}) \Big) \\ &\geq \sum_{i=1}^{n} P(\{i \text{ is isolated in } \mathcal{G}(n, r(n))\}) \\ &\quad - \sum_{i=1}^{n} \sum_{j\neq i} P(\{i \text{ and } j \text{ are isolated in } \mathcal{G}(n, r(n))\}). \end{aligned} \tag{1.8}$$

Neglecting edge effects, we get

$$\begin{aligned} P(\{i \text{ is isolated in } \mathcal{G}(n, r(n))\}) &\sim (1 - \pi r^2(n))^{n-1}, \\ P(\{i \text{ and } j \text{ isolated in } \mathcal{G}(n, r(n))\}) &\sim (4\pi r^2(n) - \pi r^2(n))(1 - \frac{5}{4}\pi r^2(n))^{n-2} \\ &\quad + (1 - 4\pi r^2(n))(1 - 2\pi r^2(n))^{n-2}. \end{aligned} \tag{1.9}$$

The first term on the RHS above takes into account the case where j is at a distance between $r(n)$ and $2r(n)$ from i. Substituting (1.9) in (1.8), we get

$$\begin{aligned} P^{(1)}(n, r(n)) &\geq n(1 - \pi r^2(n))^{n-1} - n(n-1)\Big(3\pi r^2(n)(1 - \frac{5}{4}\pi r^2(n))^{n-2} \\ &\quad + (1 - 2\pi r^2(n))^{n-2}\Big). \end{aligned}$$

Using Lemmas 2.1 and 2.2, we get that for $\pi r^2(n) = \frac{\log n + c}{n}$, and for any fixed $\theta < 1$ and $\epsilon > 0$,

$$\begin{aligned} P^{(1)}(n, r(n)) &\geq \theta e^{-c} - n(n-1)\Big(3\pi r^2(n) e^{-\frac{5}{4}(n-2)\pi r^2(n)} + e^{-2(n-2)\pi r^2(n)}\Big) \\ &\geq \theta e^{-c} - (1+\epsilon)e^{-2c}, \end{aligned}$$

for all $n > N(\epsilon, \theta, c)$. Since $P^{(1)}(n, r(n)) \leq P_d(n, r(n))$, we have

$$P_d(n, r(n)) \geq \theta e^{-c} - (1+\epsilon)e^{-2c}, \tag{1.10}$$

for all $n > N(\epsilon, \theta, c)$. Now, consider the case where c is a function $c(n)$ with $\lim_{n\to\infty} c(n) = \bar{c}$. Then, for any $\epsilon > 0$, $c(n) \leq \bar{c} + \epsilon$ for all $n \geq N'(\epsilon)$. Also, the probability of disconnectedness is monotone decreasing in c. Hence

$$P_d(n, r(n)) \geq \theta e^{-(\bar{c}+\epsilon)} - (1+\epsilon)e^{-2(\bar{c}+\epsilon)}.$$

for $n \geq \max\{N(\epsilon, \theta, \bar{c}+\epsilon), N'(\epsilon)\}$. Taking limits

$$\liminf_{n\to\infty} P_d(n, r(n)) \geq \theta e^{-(\bar{c}+\epsilon)} - (1+\epsilon)e^{-2(\bar{c}+\epsilon)}.$$

Since this holds for all $\epsilon > 0$ and $\theta < 1$, the result follows. □

Corollary 2.1 *Graph $\mathcal{G}(n, r(n))$ is asymptotically disconnected with positive probability if $\pi r^2(n) = \frac{\log n + c(n)}{n}$ and $\limsup_n c(n) < +\infty$.*

3 Sufficient Condition on $r(n)$ for Connectivity

In order to derive a lower bound on $r(n)$ so as to ensure asymptotic connectivity in $\mathcal{G}(n, r(n))$, we make use of some results from *continuum percolation* (Meester and Roy (1996)). In percolation theory, nodes are assumed to be distributed with Poisson intensity λ in $\Re^2$ (results are in fact available for more general cases, see Meester and Roy (1996)). As in $\mathcal{G}(n, r(n))$, two nodes are connected to each other if the distance between them is less than $r(\lambda)$. Let $\mathcal{G}^{\text{Poisson}}(\lambda, r(\lambda))$ denote the resultant (infinite) graph. Also, let $q_k(\lambda, r(\lambda))$ be the probability that the node at the origin is a part of an order-k component. Of course for this to make sense, the node distribution process is conditioned on the origin having a node. Then, $(1 - \sum_{k=1}^{\infty} q_k(\lambda, r(\lambda))) =: q_\infty(\lambda, r(\lambda))$ gives the probability that the origin is connected to an infinite-order component. It can be shown that almost surely $\mathcal{G}^{\text{Poisson}}(\lambda, r(\lambda))$ has at most one infinite-order component for each $\lambda \geq 0$ (Theorem 6.3 of Meester and Roy (1996)). Furthermore, the following is true (Propositions 6.4-6.6 of Meester and Roy (1996))

$$\lim_{\lambda\to\infty} \frac{1}{q_1(\lambda, r(\lambda))} \sum_{k=1}^{\infty} q_k(\lambda, r(\lambda)) = 1. \tag{1.11}$$

Hence, as $\lambda \to \infty$, almost surely the origin in $\mathcal{G}^{\text{Poisson}}(\lambda, r(\lambda))$ lies in either an infinite-order component or an order-1 component (i.e., it is isolated). Now, our original problem concerning a fixed number of nodes n in the unit-area disc $\mathcal{D}$ can be approximated by regarding that process as the restriction to $\mathcal{D}$ of the Poisson process on $\Re^2$ with $\lambda = n$. Let the graph obtained by restricting $\mathcal{G}^{\text{Poisson}}(n, r(n))$ to $\mathcal{D}$ be denoted by $\mathcal{G}_{\mathcal{D}}^{\text{Poisson}}(n, r(n))$. Then, by the above observation, the probability that $\mathcal{G}_{\mathcal{D}}^{\text{Poisson}}(n, r(n))$ is disconnected, denoted by $P_d^{\text{Poisson}}(n, r(n))$, is asymptotically the same as the probability that it has at least one isolated node, denoted by $P^{\text{Poisson};(1)}$ $(n, r(n))$. Although $\mathcal{G}_{\mathcal{D}}^{\text{Poisson}}(n, r(n))$ has a Poisson(n) number of nodes in $\mathcal{D}$, the difference between $\mathcal{G}_{\mathcal{D}}^{\text{Poisson}}(n, r(n))$ and $\mathcal{G}(n, r(n))$ is negligible for large n. This is made precise below.

Lemma 3.1 *If* $\pi r^2(n) = \frac{\log n + c(n)}{n}$, *then*

$$\limsup_{n\to\infty} P^{\text{Poisson};(1)}(n, r(n)) \le e^{-c}, \tag{1.12}$$

where $c = \lim_{n\to\infty} c(n)$.

Proof: Note that since $e^{-n}\frac{n^j}{j!}$ is the probability that $\mathcal{G}_{\mathcal{D}}^{\text{Poisson}}(n, r(n))$ has j nodes, and defining a graph with 0 nodes to be connected, we have

$$P^{\text{Poisson};(1)}(n, r(n)) = \sum_{j=1}^{\infty} P^{(1)}(j, r(n)) e^{-n}\frac{n^j}{j!}. \tag{1.13}$$

Let $E_1(j, r(n))$ denote the expected number of order-1 components in $\mathcal{G}(j,$ $r(n))$. Then

$$\begin{aligned} P^{(1)}(j, r(n)) &\le E_1(j, r(n)) \\ &= E[\sum_{i=1}^{j} I(i \text{ is isolated in } \mathcal{G}(j, r(n)))] \\ &= jP(\{j \text{ is isolated in } \mathcal{G}(j, r(n))\}) \\ &\sim j(1-\pi r^2(n))^{j-1}. \end{aligned} \tag{1.14}$$

Substituting (1.14) in (1.13), we get

$$\begin{aligned} P^{\text{Poisson};(1)}(n, r(n)) &\le \sum_{j=1}^{\infty} j(1-\pi r^2(n))^{j-1} e^{-n}\frac{n^j}{j!} \\ &= n\sum_{j=0}^{\infty}(1-\pi r^2(n))^j e^{-n}\frac{n^j}{j!} \\ &= ne^{-n\pi r^2(n)}, \end{aligned} \tag{1.15}$$

from which the result follows. □

The following must be a known fact though we are not aware of any reference for it.

Lemma 3.2 *For all $\epsilon > 0$ and sufficiently large n*

$$\sum_{j=1}^{n} e^{-n}\frac{n^j}{j!} \geq (\frac{1}{2} - \epsilon). \tag{1.16}$$

We are now ready to give a sufficient condition on $r(n)$ for asymptotic connectivity in $\mathcal{G}(n, r(n))$.

Theorem 3.1 *If $\pi r^2(n) = \frac{\log n + c(n)}{n}$ and $\lim_{n\to\infty} c(n) = c$, then*

$$\limsup_{n\to\infty} P_d(n, r(n)) \leq 4e^{-c}. \tag{1.17}$$

Proof: By (1.11) and the observation made thereafter, we get that, for any $\epsilon > 0$ and for all sufficiently large n,

$$P_d^{\text{Poisson}}(n, r(n)) \leq (1+\epsilon) P^{\text{Poisson};(1)}(n, r(n)). \tag{1.18}$$

Note that

$$P_d^{\text{Poisson}}(n, r(n)) = \sum_{j=1}^{\infty} P_d(j, r(n)) e^{-n}\frac{n^j}{j!}. \tag{1.19}$$

For a fixed range $r = r(n)$, we have

$$P_d(k, r) \leq P(\{\text{node } k \text{ is isolated in } \mathcal{G}(k, r)\}) + P_d(k-1, r).$$

which after recursion gives, that for $0 < j < n$,

$$\begin{aligned} P_d(n, r(n)) &\leq \sum_{k=j+1}^{n} P(\{\text{node } k \text{ is isolated in } \mathcal{G}(k, r(n))\}) + P_d(j, r(n)) \\ &\leq \sum_{k=j+1}^{n} (1 - \pi r^2(n))^{k-1} + P_d(j, r(n)) \\ &\leq \frac{(1-\pi r^2(n))^j}{\pi r^2(n)} + P_d(j, r(n)). \end{aligned} \tag{1.20}$$

Substituting (1.20) in (1.19), we get

$$\begin{aligned} P_d^{\text{Poisson}}(n, r(n)) &\geq P_d(n, r(n)) \sum_{j=1}^{n} e^{-n}\frac{n^j}{j!} - \sum_{j=1}^{n-1} \frac{(1-\pi r^2(n))^{j-1}}{\pi r^2(n)} e^{-n}\frac{n^j}{j!} \\ &\geq P_d(n, r(n))(\frac{1}{2} - \epsilon) - \frac{e^{-n\pi r^2(n)}}{\pi r^2(n)}, \end{aligned} \tag{1.21}$$

where we have used Lemma 3.2. Using (1.18), we get

$$P_d(n, r(n)) \leq 2(1+4\epsilon)\left[P^{\text{Poisson};(1)}(n, r(n)) + \frac{e^{-n\pi r^2(n)}}{\pi r^2(n)}\right]. \tag{1.22}$$

For the given $\pi r^2(n) = \frac{\log n + c(n)}{n}$, from Lemma 3.1, we get that, for any fixed $\epsilon > 0$, the following holds for all sufficiently large n

$$P_d(n, r(n)) \leq 2(1+4\epsilon)\left[e^{-c(n)} + \frac{e^{-c(n)}}{\log n + c(n)}\right].$$

Thus, since $\epsilon > 0$ is arbitrary,

$$\limsup_{n\to\infty} P_d(n, r(n)) \leq 2e^{-c}. \tag{1.23}$$

$\square$

The following is an obvious consequence of Theorem 3.1.

Corollary 3.1 *Graph $\mathcal{G}(n, r(n))$ is asymptotically connected with probability one for $\pi r^2(n) = \frac{\log n + c(n)}{n}$ if $c(n) \to +\infty$.*

Combining Corollaries 2.1 and 3.1, we get the main result of the paper.

Theorem 3.2 *Graph $\mathcal{G}(n, r(n))$, with $\pi r^2(n) = \frac{\log n + c(n)}{n}$ is connected with probability one as $n \to \infty$ if and only if $c(n) \to +\infty$.*

4 Concluding Remarks

We have derived the *critical* range of nodes placed randomly in a disc of unit area, for the resulting network to be connected with probability one as the number of nodes tends to infinity (cf. Theorem 3.2). One can consider the following extensions of the problem discussed in this paper:

- Our lower and upper bounds on $P_d(n, r(n))$ are not tight. A more refined argument may lead to bounds which hold for all n. In particular, we believe that for $\pi r^2(n) = \frac{\log n + c(n)}{n}$, $P_d(n, r(n)) \to 1$ if $c(n) \to -\infty$.

- Consider the following generalization of the problem: Even if a node has another node in its range, it can communicate with that node with probability $p(n), 0 \leq p(n) \leq 1$. The quantity $p(n)$ can be regarded as the reliability of a link, and is tantamount to Bernoulli deletion of edges in $\mathcal{G}(n, r(n))$. Our conjecture is that Theorem 3.2 is true with $\pi r^2(n)$ replaced by $\pi r^2(n)p(n)$. This conjecture holds for at least two special cases: $\pi r^2(n) \equiv 4$ (i.e., range of each node includes $\mathcal{D}$) and $p(n)$ arbitrary in $[0,1]$ (Theorem VII.3 in Bollobás (1985)), and $r(n)$ arbitrary and $p(n) \equiv 1$ (cf. Theorem 3.2). As in the proof of Theorem 3.1, continuum percolation theory results can be used to obtain sufficient conditions on $\pi r^2(n)p(n)$. Clearly, Theorem 2.1 still holds. However, stronger necessary conditions need to be worked out.

- A much harder problem to analyze is when nodes are not placed independently in the disc $\mathcal{D}$. For example, nodes may be placed in clusters, with a specified probability distribution on the size of a cluster.

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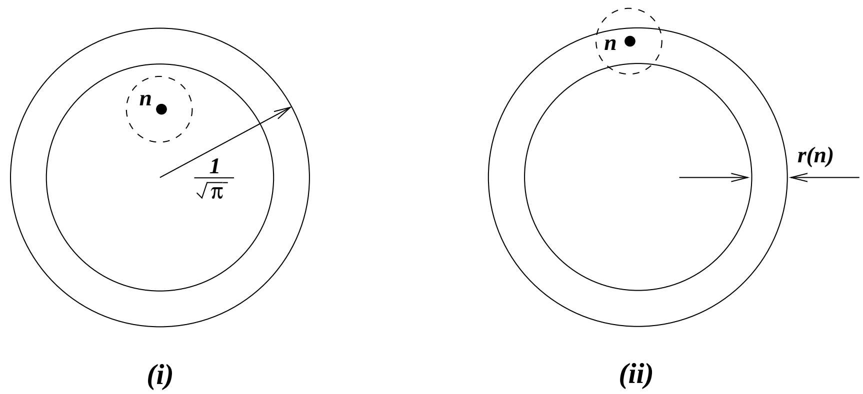

FIGURE 1. Two cases for evaluating $P(\{n \text{ is isolated in } \mathcal{G}(n, r(n)\})$.

1 Appendix

Here we give the complete proofs of the theorems given in the main body of the paper, taking the edge effects into account.

Proof of Theorem 2.1:
As before, we first study the case where $\pi r^2(n) = \frac{\log n + c}{n}$ for a fixed c. Consider $P^{(1)}(n, r(n))$, the probability that $\mathcal{G}(n, r(n))$ has *at least* one order-1 component. Then, as argued in (1.8), we have

$$\begin{aligned} P^{(1)}(n, r(n)) \quad \geq \quad & \sum_{i=1}^{n} P(\{i \text{ is isolated in } \mathcal{G}(n, r(n))\}) \\ & - \sum_{i=1}^{n} \sum_{j \neq i} P(\{i \text{ and } j \text{ are isolated in } \mathcal{G}(n, r(n))\}). \quad (.24) \end{aligned}$$

Now, let us consider each sum in (.24) separately. For this purpose, define the notation

$$\begin{aligned} N^{(1)}(\mathcal{G}) \quad &:= \quad \{i \in \mathcal{G} : \ i \text{ is an isolated node in } \mathcal{G}\}, \\ \mathcal{D}^o \quad &:= \quad \{x \in \mathcal{D} : \|x\| \leq \frac{1}{\sqrt{\pi}} - r(n)\}, \\ \partial\mathcal{D} \quad &:= \quad \mathcal{D} - \mathcal{D}^o. \qquad (.25) \end{aligned}$$

Then, as illustrated in Figure 1, we need to consider two cases to evaluate the probability that node n is isolated, namely: When $x_n \in \mathcal{D}^o$ (recall that x_n is the position of node n in $\mathcal{D}$), and when $x_n \in \partial\mathcal{D}$. To obtain a lower bound, we consider only the first case, i.e.,

$$\sum_{i=1}^{n} P(\{i \text{ is isolated in } \mathcal{G}(n, r(n))\}) = nP(\{n \text{ is isolated in } \mathcal{G}(n, r(n)\})$$

$$\begin{aligned} &\geq nP(\{x_n \in \mathcal{D}_o,\, n \in N^{(1)}(\mathcal{G}(n, r(n)))\}) \\ &= n\pi(\frac{1}{\sqrt{\pi}} - r(n))^2(1 - \pi r^2(n))^{n-1}. \end{aligned}$$

Using Lemma 2.2, we see that for any $\theta < 1$, when n is sufficiently large,

$$\sum_{i=1}^{n} P(\{i \text{ is isolated in } \mathcal{G}(n, r(n)\}) \geq \theta e^{-c}. \tag{.26}$$

Next, consider the second sum in (.24), which in the notation of (.25) can be written as

$$\begin{aligned} &\sum_{i=1}^{n}\sum_{j\neq i} P(\{i, j \in N^{(1)}(\mathcal{G}(n, r(n)))\}) \\ &= \sum_{i=1}^{n}\sum_{j\neq i}\Big(P(\{i, j \in N^{(1)}(\mathcal{G}(n, r(n))),\text{ and } x_i \text{ or } x_j \in \partial\mathcal{D}\}) \\ &\quad + P(\{i, j \in N^{(1)}(\mathcal{G}(n, r(n))),\text{ and } x_i, x_j \in \mathcal{D}^o\})\Big) \\ &\leq n(n-1)\Big(2P(\{n, n-1 \in N^{(1)}(\mathcal{G}(n, r(n))),\text{ and } x_n \in \partial\mathcal{D}\}) \\ &\quad + P(\{n, n-1 \in N^{(1)}(\mathcal{G}(n, r(n))),\text{ and } x_n, x_{n-1} \in \mathcal{D}^o\})\Big). \end{aligned} \tag{.27}$$

The first term can be written as

$$\begin{aligned} &2n(n-1)P(\{n, n-1 \in N^{(1)}(\mathcal{G}(n, r(n))),\text{ and } x_n \in \partial\mathcal{D}\}) \\ &\quad = 2n(n-1)\cdot P(\{n \in N^{(1)}(\mathcal{G}(n, r(n))),\text{ and } x_n \in \partial\mathcal{D}\})\cdot \\ &\quad P(\{n-1 \in N^{(1)}(\mathcal{G}(n, r(n)))\Big| n \in N^{(1)}(\mathcal{G}(n, r(n))), x_n \in \partial\mathcal{D}\}). \end{aligned} \tag{.28}$$

Now $nP(\{n \in N^{(1)}(\mathcal{G}(n, r(n))),\text{ and } x_n \in \partial\mathcal{D}\})$ can be evaluated using Figure 2, to give

$$\begin{aligned} nP(\{n \in N^{(1)}(\mathcal{G}(n, r(n))),\text{ and } x_n \in \partial\mathcal{D}\}) &\leq n\int_0^{r(n)}\Bigg(1 - \\ \left(\pi - \cos^{-1}\frac{y}{r(n)}\right) r^2(n) + \mathcal{E}(y)\Bigg)^{n-1} & 2\pi(\frac{1}{\sqrt{\pi}} - y)dy, \end{aligned} \tag{.29}$$

where

$$\begin{aligned} \mathcal{E}(y) &\leq 2r(n)\cdot\left(\frac{1}{\sqrt{\pi}} - \sqrt{\frac{1}{\pi} - r^2(n)}\right) \\ &\leq 2r(n)\cdot\left(\frac{1}{\sqrt{\pi}} - \frac{1}{\sqrt{\pi}}\left(1 - \pi r^2(n)\right)\right) \\ &= 2\sqrt{\pi}r^3(n). \end{aligned} \tag{.30}$$

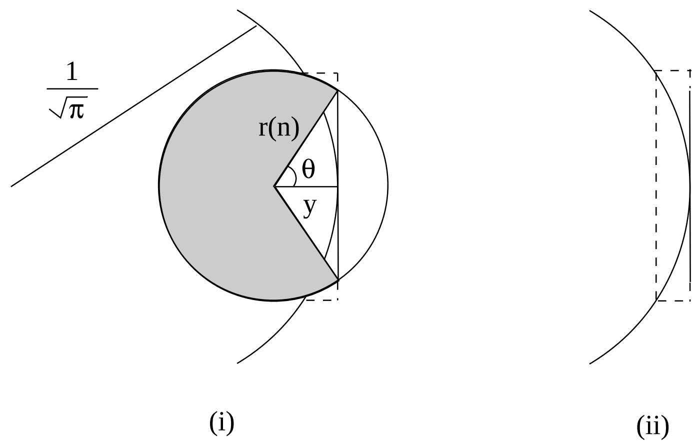

FIGURE 2. Evaluating $nP(\{n \in N^{(1)}(\mathcal{G}(n,r(n))),$ and $x_n \in \partial\mathcal{D}\})$: (i) Shaded area gives a lower bound on the area which nodes $1, 2, \ldots, n-1$ should not lie in for node n to be isolated, when n is at a distance of y from the boundary of $\mathcal{D}$, and (ii) Area within the rectangle is an upper bound on the error $\mathcal{E}(y)$ due to approximation of the portion of the disc $\mathcal{D}$ within the range of node n by its tangent.

Substituting (.30) in (.29), and changing the variable to $\theta = \cos^{-1}\frac{y}{r(n)}$, we get

$$\begin{aligned}
&nP(\{n \in N^{(1)}(\mathcal{G}(n,r(n))), \text{ and } x_n \in \partial\mathcal{D}\}) \\
&\quad\leq n\int_0^{\frac{\pi}{2}} \left(1-(\pi-\theta)r^2(n)+2\sqrt{\pi}r^3(n)\right)^{n-1} 2\sqrt{\pi}r(n)\sin\theta d\theta \\
&\quad\leq n\cdot 2\sqrt{\pi}r(n)\int_0^{\frac{\pi}{2}} e^{-(n-1)((\pi-\theta)r^2(n)-2\sqrt{\pi}r^3(n))}\frac{e^{j\theta}-e^{-j\theta}}{2j}d\theta \\
&\quad= 2\sqrt{\pi}nr(n)e^{-(n-1)(\pi r^2(n)-2\sqrt{\pi}r^3(n))}\cdot \\
&\qquad\frac{e^{(n-1)\frac{\pi}{2}r^2(n)}(n-1)r^2(n)+1}{((n-1)r^2(n))^2+1}. \qquad (.31)
\end{aligned}$$

For the given $\pi r^2(n) = \frac{\log n + c}{n}$, we thus have

$$\begin{aligned}
&nP(\{n \in N^{(1)}(\mathcal{G}(n,r(n))), \text{ and } x_n \in \partial\mathcal{D}\}) \\
&\quad\leq \frac{4\pi n e^{-(n-1)(\frac{\pi}{2}r^2(n)-2\sqrt{\pi}r^3(n))}}{(n-1)\sqrt{\pi}r(n)} \\
&\quad\leq \frac{4(1+\epsilon)\pi e^{-\frac{c}{2}}}{\sqrt{\log n}}, \qquad (.32)
\end{aligned}$$

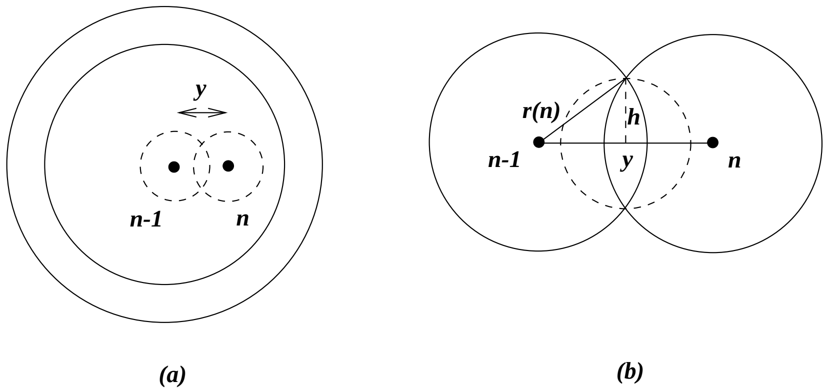

FIGURE 3. Evaluating $P(\{n, (n-1) \in N^{(1)}(\mathcal{G}(n, r(n))),$ and $x_n, x_{n-1} \in \mathcal{D}^o\})$. Note that $h^2 = r^2(n) - \left(\frac{y}{2}\right)^2$.

for any $\epsilon > 0$ and sufficiently large n. The remaining factor in (.28) can be evaluated as

$$\begin{aligned}
&2(n-1)P(\{n-1 \in N^{(1)}(\mathcal{G}(n,r(n)))\Big| n \in N^{(1)}(\mathcal{G}(n,r(n))), x_n \in \partial\mathcal{D}\}) \\
&\quad = \quad 2(n-1)\Big(P(\{n-1 \in N^{(1)}(\mathcal{G}(n,r(n))), x_{n-1} \in \partial\mathcal{D}\Big| \\
&\qquad\quad n \in N^{(1)}(\mathcal{G}(n,r(n))), x_n \in \partial\mathcal{D}\}) + P(\{n-1 \in N^{(1)}(\mathcal{G}(n,r(n))), \\
&\qquad\quad x_{n-1} \in \mathcal{D}^o\Big| n \in N^{(1)}(\mathcal{G}(n,r(n))), x_n \in \partial\mathcal{D}\})\Big) \\
&\quad \leq \quad 2(n-1)\Big(2\sqrt{\pi}r(n)(1-\pi r^2(n) + 2\mathcal{E}(0))^{n-2} \\
&\qquad\quad + \ (1 - \frac{3}{2}\pi r^2(n) + \mathcal{E}(0))^{n-2}\Big), \qquad (.33)
\end{aligned}$$

where $\mathcal{E}(\cdot)$ is defined in (.30). For the given $\pi r^2(n) = \frac{\log n + c}{n}$, we thus have

$$\begin{aligned}
&2(n-1)P(\{n-1 \in N^{(1)}(\mathcal{G}(n,r(n)))\Big| n \in N^{(1)}(\mathcal{G}(n,r(n))), x_n \in \partial\mathcal{D}\}) \\
&\qquad\qquad \leq 4(1+\epsilon)\sqrt{\frac{\log n}{n}}, \qquad (.34)
\end{aligned}$$

for any $\epsilon > 0$ and all sufficiently large n. Substituting (.32) and (.34) in (.28), we get

$$\begin{aligned}
&2n(n-1)P(\{n, n-1 \in N^{(1)}(\mathcal{G}(n,r(n))), \text{ and } x_n \in \partial\mathcal{D}\}) \\
&\quad \leq \quad \frac{4(1+\epsilon)\pi e^{-\frac{c}{2}}}{\sqrt{\log n}} \cdot 4(1+\epsilon)\sqrt{\frac{\log n}{n}}
\end{aligned}$$

$$\leq \quad \frac{16(1+\epsilon')\pi e^{-\frac{c}{2}}}{\sqrt{n}}, \tag{.35}$$

for any $\epsilon' > 0$ and all sufficiently large n. The second term in (.27) is (as illustrated in Figure 3),

$$\begin{aligned}
&n(n-1)P(\{n,(n-1) \in N^{(1)}(\mathcal{G}(n,r(n))), \text{ and } x_n, x_{n-1} \in \mathcal{D}^o\}) \\
&\leq \quad n(n-1)P(\{x_n \in \mathcal{D}^o\}) \cdot \Big[P(\{r(n) < |x_n - x_{n-1}| \leq 2r(n), |x_i - x_j| \\
&\qquad > r(n), 1 \leq i \leq n-2; j = n, n-1\}\Big| x_n \in \mathcal{D}^o) + P(\{2r(n) < |x_n \\
&\qquad - x_{n-1}|, |x_i - x_j| > r(n), 1 \leq i \leq n-2; j = n, n-1\}\Big| x_n \in \mathcal{D}^o)\Big] \\
&\leq \quad n(n-1)\pi(\frac{1}{\sqrt{\pi}} - r(n))^2 \cdot \\
&\qquad \left[\int_{r(n)}^{2r(n)} \left(1 - 2\pi r^2(n) + \pi\left(r^2(n) - \frac{y^2}{4}\right)\right)^{n-2} 2\pi y dy + \right. \\
&\qquad \left. + \left(1 - \pi r^2(n)\right)\left(1 - 2\pi r^2(n)\right)^{n-2}\right] \\
&\leq \quad n(n-1)\left[\int_{r(n)}^{2r(n)} e^{-(n-2)\pi(r^2(n)+\frac{y^2}{4})} 2\pi y dy + (1 - 2\pi r^2(n))^{n-2}\right] \\
&\leq \quad n(n-1)\left[e^{-(n-2)\pi r^2(n)} \frac{4}{n-2} e^{-\frac{(n-2)\pi y^2}{4}}\Big|_{2r(n)}^{r(n)} + e^{-(n-2)2\pi r^2(n)}\right] \\
&\leq \quad n(n-1)\left[\frac{4}{n-2} e^{-(n-2)\frac{5}{4}\pi r^2(n)} + e^{-(n-2)2\pi r^2(n)}\right] \\
&\leq \quad n(n-1)(1+\epsilon'')e^{-(n-2)2\pi r^2(n)} \\
&\leq \quad (1+\epsilon')e^{-2c},
\end{aligned} \tag{.36}$$

for any $\epsilon' > 0$, the given $\pi r^2(n) = \frac{\log n + c}{n}$ and all sufficiently large n. Substituting (.35) and (.36) in (.27), we get

$$\begin{aligned}
&\sum_{i=1}^{n} \sum_{j \neq i}^{n} P(\{i \text{ and } j \text{ are isolated in } \mathcal{G}(n,r(n))\}) \\
&\qquad \leq \quad \frac{16\pi(1+\epsilon'')}{\sqrt{n}} e^{-\frac{c}{2}} + (1+\epsilon')e^{-2c} \\
&\qquad \leq \quad (1+\epsilon)e^{-2c}.
\end{aligned} \tag{.37}$$

for any $\epsilon > 0$ and all sufficiently large n. Substituting (.26) and (.37) in (.24), we get

$$P^{(1)}(n, r(n)) \geq \theta e^{-c} - (1+\epsilon)e^{-2c},$$

for all $n > N(\epsilon, \theta, c)$. Since $P^{(1)}(n, r(n)) \leq P_d(n, r(n))$, we have

$$P_d(n, r(n)) \geq \theta e^{-c} - (1+\epsilon)e^{-2c}, \tag{.38}$$

for all $n > N(\epsilon, \theta, c)$. Now, consider the case where c is a function $c(n)$ with $\lim_{n\to\infty} c(n) = \bar{c}$. Then, for any $\epsilon > 0$, $c(n) \leq \bar{c} + \epsilon$ for all $n \geq N'(\epsilon)$. Also, the probability of disconnectedness is monotone decreasing in c. Hence

$$P_d(n, r(n)) \geq \theta e^{-(\bar{c}+\epsilon)} - (1+\epsilon)e^{-2(\bar{c}+\epsilon)}.$$

for $n \geq \max\{N(\epsilon, \theta, \bar{c}+\epsilon), N'(\epsilon)\}$. Taking limits

$$\liminf_{n\to\infty} P_d(n, r(n)) \geq \theta e^{-(\bar{c}+\epsilon)} - (1+\epsilon)e^{-2(\bar{c}+\epsilon)}.$$

Since this holds for all $\epsilon > 0$ and $\theta < 1$, the result follows. □

Proof of Lemma 3.1:
As before,

$$P^{\text{Poisson};(1)}(n, r(n)) = \sum_{j=1}^{\infty} P^{(1)}(j, r(n)) e^{-n} \frac{n^j}{j!}. \tag{.39}$$

Let $E_1(j, r(n))$ denote the expected number of order-1 components in $\mathcal{G}(j, r(n))$. Then

$$\begin{aligned} P^{(1)}(j, r(n)) &\leq E_1(j, r(n)) \\ &= E[\sum_{i=1}^{j} I(i \text{ is isolated in } \mathcal{G}(j, r(n)))] \\ &= jP(\{j \text{ is isolated in } \mathcal{G}(j, r(n))\}). \end{aligned} \tag{.40}$$

Using the definitions of $N^{(1)}(\mathcal{G}), \mathcal{D}^o$ and $\partial\mathcal{D}$ given in (.25), we can write

$$\begin{aligned} &P(\{j \text{ is isolated in } \mathcal{G}(j, r(n))\}) \\ &\quad = P(\{j \in N^{(1)}(\mathcal{G}(j, r(n))) \text{ and } x_j \in \mathcal{D}^o\}) \\ &\qquad + P(\{j \in N^{(1)}(\mathcal{G}(j, r(n))) \text{ and } x_j \in \partial\mathcal{D}\}). \end{aligned} \tag{.41}$$

¿From (.31) and (.41) , we get

$$\begin{aligned} P^{(1)}(j, r(n)) &\leq jP(\{j \text{ is isolated in } \mathcal{G}(j, r(n))\}) \\ &\leq j\pi \left(\frac{1}{\sqrt{\pi}} - r(n)\right)^2 \left(1 - \pi r^2(n)\right)^{j-1} + 2\sqrt{\pi} j r(n) \cdot \\ &\qquad e^{-(j-1)f_1(r(n))} \frac{e^{(j-1)f_2(r(n))}(j-1)r^2(n) + 1}{\left((j-1)r^2(n)\right)^2 + 1}, \end{aligned} \tag{.42}$$

where $f_1(r) = \pi r^2 - 2\sqrt{\pi} r^3$, and $f_2(r) = \pi r^2/2$. From (.39) and (.42), we get

$$P^{\text{Poisson};(1)}(n, r(n)) \leq \sum_{j=1}^{\infty} jP(\{j \text{ is isolated in } \mathcal{G}(j, r(n))\})e^{-n}\frac{n^j}{j!} \quad (.43)$$

$$\begin{aligned}
\leq \quad & \sum_{j=1}^{\infty} j(1-\pi r^2(n))^{j-1}e^{-n}\frac{n^j}{j!} + 2\sqrt{\pi}r(n)e^{-n}\frac{n}{1!} \\
& + 2\sqrt{\pi}r(n)\sum_{j=2}^{\infty} j\left[\frac{e^{-(j-1)(f_1(r(n))-f_2(r(n)))}}{(j-1)r^2(n)} + \frac{e^{-(j-1)f_1(r(n))}}{((j-1)r^2(n))^2}\right]e^{-n}\frac{n^j}{j!} \\
\leq \quad & n\sum_{j=0}^{\infty}(1-\pi r^2(n))^j e^{-n}\frac{n^j}{j!} + 2\sqrt{\pi}r(n)ne^{-n} \\
& + \frac{2\sqrt{\pi}}{r(n)}\sum_{j=2}^{\infty}\frac{j}{j-1}\frac{e^{-(j-1)(f_1(r(n))-f_2(r(n)))}n^j}{j!}e^{-n} \\
& + \frac{2\sqrt{\pi}}{r^3(n)}\sum_{j=2}^{\infty}\frac{j}{j-1}\frac{j+1}{j-1}\frac{e^{-(j-1)f_1(r(n))}n^j}{(j+1)!}e^{-n} \\
\leq \quad & ne^{-n\pi r^2(n)} + 2\sqrt{\pi}r(n)ne^{-n} + \frac{2\sqrt{\pi}}{r(n)}2e^{f_1(r(n))-f_2(r(n))} \cdot \\
& \cdot e^{ne^{-(f_1(r(n))-f_2(r(n)))}}e^{-n} + \frac{2\sqrt{\pi}}{r^3(n)}\frac{2\cdot 3e^{2f_1(r(n))}}{n}e^{ne^{-f_1(r(n))}}e^{-n} \\
\leq \quad & ne^{-n\pi r^2(n)} + 2\sqrt{\pi}r(n)ne^{-n} + \frac{4\sqrt{\pi}}{r(n)}e^{-(n-1)(f_1(r(n))-f_2(r(n)))} \cdot \\
& \cdot e^{\frac{n(f_1(r(n))-f_2(r(n)))^2}{2}} + \frac{12\sqrt{\pi}}{nr^3(n)}e^{-(n-2)f_1(r(n))}e^{\frac{nf_1^2(r(n))}{2}}, \qquad (.44)
\end{aligned}$$

where we have used $e^{-x} \leq 1 - x + \frac{x^2}{2}$. For the given $\pi r^2(n) = \frac{\log n + c(n)}{n}$, we thus have

$$\begin{aligned}
P^{\text{Poisson};(1)}(n, r(n)) \quad & \leq \quad e^{-c(n)} + 2\sqrt{n(\log n + c(n))}e^{-n} + \frac{4\pi(1+\epsilon)e^{-c(n)}}{\sqrt{\log n + c(n)}} \\
& + \frac{12\pi^2(1+\epsilon)}{\sqrt{n(\log n + c(n))^3}}e^{-c(n)}, \qquad (.45)
\end{aligned}$$

for any $\epsilon > 0$ and all sufficiently large n. The result follows. □

Proof of Lemma 3.2:

By Chebyshev's inequality, we have that for any α

$$\sum_{j=n+n^{\alpha}+1}^{\infty} e^{-n}\frac{n^j}{j!} \leq \frac{n}{n^{2\alpha}}.$$

Let $\alpha = \frac{1}{2} + \epsilon$ for some $\epsilon > 0$, then

$$\sum_{j=n+n^\alpha+1}^{\infty} e^{-n} \frac{n^j}{j!} \le \frac{1}{n^{2\epsilon}}. \tag{.46}$$

Also,

$$\begin{aligned}
\sum_{j=n+1}^{n+n^\alpha} e^{-n} \frac{n^j}{j!} &= \sum_{j=1}^{n^\alpha} e^{-n} \frac{n^{n+j}}{(n+j)!} \\
&= \sum_{j=1}^{n^\alpha} e^{-n} \frac{n^{n-j-1}}{(n-j-1)!} \cdot \frac{1}{\prod_{i=0}^{j} \left(1 - \left(\frac{i}{n}\right)^2\right)} \\
&\le \left(\sum_{j=1}^{n^\alpha} e^{-n} \frac{n^{n-j-1}}{(n-j-1)!} \right) \cdot \\
&\qquad \max_{1 \le j \le n^\alpha} \frac{1}{\prod_{i=0}^{j} \left(1 - \left(\frac{i}{n}\right)^2\right)}.
\end{aligned} \tag{.47}$$

Now,

$$\begin{aligned}
\max_{1 \le j \le n^\alpha} \frac{1}{\prod_{i=0}^{j} \left(1 - \left(\frac{i}{n}\right)^2\right)} &\le \max_{1 \le j \le n^\alpha} \frac{1}{1 - \sum_{i=1}^{j} \left(\frac{i}{n}\right)^2} \\
&= \frac{1}{1 - \sum_{i=1}^{n^\alpha} \left(\frac{i}{n}\right)^2} \\
&= \frac{1}{1 - \frac{n^\alpha (n^\alpha+1)(2n^\alpha+1)}{6n^2}} \\
&\le \frac{1}{1 - \frac{(1+\epsilon')n^{3\epsilon - \frac{1}{2}}}{3}},
\end{aligned} \tag{.48}$$

for the chosen $\alpha = \frac{1}{2} + \epsilon$, any $\epsilon' > 0$ and all sufficiently large n. Substituting (.48) in (.47), we get

$$\begin{aligned}
\sum_{j=n+1}^{n+n^\alpha} e^{-n} \frac{n^j}{j!} &\le \left(\sum_{j=1}^{n^\alpha} e^{-n} \frac{n^{n-j-1}}{(n-j-1)!} \right) \cdot \left(1 + \frac{2(1+\epsilon')n^{3\epsilon - \frac{1}{2}}}{3} \right) \\
&\le \left(\sum_{j=1}^{n} e^{-n} \frac{n^j}{j!} \right) \cdot \left(1 + \frac{2(1+\epsilon')n^{3\epsilon - \frac{1}{2}}}{3} \right).
\end{aligned} \tag{.49}$$

¿From (.46), (.49) and the fact that $\sum_{j=0}^{\infty} e^{-n} n^j/j! = 1$, we get that

$$\begin{aligned}\sum_{j=1}^{n} e^{-n}\frac{n^j}{j!} &\geq \frac{1 - e^{-n} - \frac{1}{n^{2\epsilon}}}{1 + 1 + \frac{2(1+\epsilon')n^{3\epsilon - \frac{1}{2}}}{3}} \\ &\geq \frac{1}{2} - \epsilon'',\end{aligned}$$

for any $\epsilon < \frac{1}{6}$, $\epsilon'' > 0$ and all sufficiently large n. □

Proof of Theorem 3.1:
By (1.11) and the observation made thereafter, we get that, for any $\epsilon > 0$ and for all sufficiently large n,

$$P_d^{\text{Poisson}}(n, r(n)) \leq (1+\epsilon) P^{\text{Poisson};(1)}(n, r(n)). \tag{.50}$$

Note that

$$P_d^{\text{Poisson}}(n, r(n)) = \sum_{j=1}^{\infty} P_d(j, r(n)) e^{-n}\frac{n^j}{j!}. \tag{.51}$$

For a fixed range $r = r(n)$, we have

$$P_d(k, r) \leq P(\{\text{node } k \text{ is isolated in } \mathcal{G}(k, r)\}) + P_d(k-1, r).$$

which after recursion gives, that for $0 \leq j < n$

$$\begin{aligned}P_d(n, r(n)) &\leq \sum_{k=j+1}^{n} P(\{\text{node } k \text{ is isolated in } \mathcal{G}(k, r(n))\}) \\ &+ P_d(j, r(n)).\end{aligned} \tag{.52}$$

Substituting (.52) in (.51), we get

$$\begin{aligned}&P_d^{\text{Poisson}}(n, r(n)) \\ &\geq P_d(n, r(n)) \sum_{j=1}^{n} e^{-n}\frac{n^j}{j!} \\ &\quad - \sum_{j=1}^{n-1}\sum_{k=j+1}^{n} P(\{k \text{ is isolated in } \mathcal{G}(k, r(n))\}) e^{-n}\frac{n^j}{j!} \\ &\geq P_d(n, r(n))(\frac{1}{2} - \epsilon) - \sum_{k=2}^{n} P(\{k \text{ is isolated in } \mathcal{G}(k, r(n))\}) \sum_{j=1}^{k-1} e^{-n}\frac{n^j}{j!} \\ &\geq P_d(n, r(n))(\frac{1}{2} - \epsilon) - \sum_{k=2}^{n} P(\{k \text{ is isolated in } \mathcal{G}(k, r(n))\}) k e^{-n}\frac{n^k}{k!},\end{aligned}$$

where we have used Lemma 3.2 and the fact that $e^{-n}\frac{n^k}{k!}$ increases with k, for $1 \leq k \leq n$. Using (.50), we get

$$\begin{aligned} P_d(n, r(n)) \quad &\leq \quad 2(1+6\epsilon)\Bigg[P^{\text{Poisson};(1)}(n, r(n)) \\ &\quad + \sum_{k=1}^{\infty} kP(\{k \text{ is isolated in } \mathcal{G}(k, r(n))\})e^{-n}\frac{n^k}{k!}\Bigg]. \end{aligned}$$

For the given $\pi r^2(n) = \frac{\log n + c(n)}{n}$, and from Lemma 3.1, and (.43) we get that for any $\epsilon > 0$,

$$P_d(n, r(n)) \leq 2(1+6\epsilon)2 \cdot (1+\epsilon')e^{-c(n)}.$$

holds for all sufficiently large n. Thus,

$$\limsup_{n\to\infty} P_d(n, r(n)) \leq 4(1+\epsilon'')e^{-c}.$$

Since ϵ'' can be made arbitrarily small, the result follows. $\square$

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