AN ADAPTIVE CONTROLLER INSPIRED BY RECENT RESULTS ON LEARNING FROM EXPERTS

P. R. KUMAR
UNIVERSITY OF ILLINOIS
DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING
AND THE COORDINATED SCIENCE LAB
1308 WEST MAIN STREET
URBANA, IL 61801, USA*

Abstract. In computational learning theory there have been some interesting developments recently on the problem of "learning from experts."

In this paper, we "adapt" the learning problem to an adaptive control formulation. What results is an adaptive controller which is reminiscent of a certainty equivalence scheme using the "posterior mean" for the parameter estimator. We show that this scheme can be analyzed in a somewhat novel way, for ideal linear systems. The analysis techniques may be of some interest to researchers in the theory of adaptive control.

Key words. Adaptive Control, Learning Theory

AMS(MOS) subject classifications.

1. Introduction. Recently, Littlestone and Warmuth [] and Cesa-Bianchi, *et al.* [] have addressed the interesting problem of learning from experts.

Here we show how these results may be "adapted" to an adaptive control framework. Briefly, we regard each parameter vector θ as giving an "expert prediction" of the next value of the output. Over time, we acquire more confidence in some "experts" and less in others. We adopt the learning scheme from [] and [] to fashion a parameter estimator. Inspired by the techniques there, we also provide a somewhat novel analysis of our adaptive controller, which may be of interest in its own right to those interested in the theory of adaptive control, e.g., techniques to establish stability and other asymptotic properties.

2. System Description. Consider a standard "ideal" linear system,

$$y(t) = \phi^T(t-1)\theta^\circ$$

where

$$\phi(t-1) := (y(t-1), \ldots, y(t-p), u(t-1), \ldots, u(t-p))^T,$$

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and

$$\theta^\circ := (a_1, \ldots, a_p, b_1, \ldots, b_p)^T .$$

Here $u(t)$ and $y(t)$ are, respectively, the input and output to the system.

We assume that θ° is in the interior of Θ, a closed sphere of unit volume, centered at the origin. (These assumptions can be generalized somewhat). We also suppose that the system is of strictly minimum phase.

Except for these assumptions, we assume that the parameter vector θ° is unknown. Our goal is to adaptively control the system in such a way that

$$\lim_{t \to \infty} y(t) = 0,$$

while $u(t)$ is kept bounded.

3. A New Adaptive Controller. Let $n(t-1) = 1 + 2\max\{1,\ \sup_\Theta \|\theta\|\}\|\phi(t-1)\|$ be a "normalization" signal. Note that

$$\frac{\|\phi(t-1)\|}{n(t-1)} \le \frac{1}{2} \text{ and } \frac{|y(t)|}{n(t-1)} \le \frac{1}{2}.$$

Let $0 < \mu < 1$. Set

$$q(0, \theta) \equiv 1, \qquad \text{for all } \theta \in \Theta, \tag{3.1}$$

and recursively define,

$$q(t, \theta) = F(t) q(t-1, \theta),$$

where $F(t)$ only needs to satisfy

$$(1-\mu)^{\frac{|y(t)-\phi^T(t-1)\theta|}{n(t-1)}} \le F(t) \le 1 - \mu \frac{|y(t) - \phi^T(t-1)\theta|}{n(t-1)}. \tag{3.2}$$

(We note that $F(t)$ is allowed to depend on past measurements, and at each t should be chosen to satisfy the bounds given above). Intuitively, one can think of $q(t, \theta)$ as our "confidence," at time t, that the value of θ° is θ. Note, however, that $q(t, \cdot)$ is "unnormalized" since $\int_\Theta q(t, \theta) d\,\theta$ need not be 1; hence $q(t, \theta)$ can be regarded as an "unnormalized" density function.

Two examples of confidence updating schemes which satisfy (3.2) are given below.

Example 1. Consider the system,

$$y(t) = \phi^T(t-1)\theta^\circ + w(t),$$

where $\theta^\circ \sim U(\Theta)$, i.e., uniformly distributed over Θ. The term $w(t)$ represents an additive noise. Assume that $\{w(t)\}$ is a sequence of independent random variables with density,

$$\begin{aligned} p_{w(t)}(w) &= 1-\mu\frac{|w|}{n(t-1)}, \text{ for } \frac{-1+\sqrt{1-\mu/n}}{\mu} \leq \frac{w}{n(t-1)} \leq \frac{1-\sqrt{1-\mu/n}}{\mu}, \\ &= 0, \qquad\qquad\qquad \text{otherwise.} \end{aligned}$$

Then the unnormalized posterior density for θ° is given by,

$$q(t,\theta) = q(t-1,\theta) - \mu q(t-1,\theta)\frac{|y(t)-\phi^T(t-1)\theta|}{n(t-1)}.$$

This is the expression for the upper bound in (3.2). Thus, the "confidence" $q(t,\theta)$ has a Bayesian interpretation. □

Example 2. Consider the same system as in Example 1, except that

$$p_{w(t)}(w) = \frac{n(t-1)(1-\mu)^{\frac{|w|}{n(t-1)}}}{-2\ln(1-\mu)}.$$

Then

$$q(t,\theta) = q(t-1,\theta)(1-\mu)^{\frac{|y(t)-\phi^T(t-1)\theta|}{n(t-1)}}$$

is the recursion for the unnormalized posterior density. It corresponds to the lower bound in (3.2). □

Let us define

$$\hat{\theta}(t) := \frac{\int_\Theta \theta q(t,\theta)\mathrm{d}\,\theta}{\int_\Theta q(t,\theta)\mathrm{d}\,\theta}. \tag{3.3}$$

It can be regarded as the "mean value" of our confidence distribution.

We will adopt a "certainty equivalent" approach, and apply a control $u(t)$ which results in

$$\phi^T(t)\hat{\theta}(t) = 0. \tag{3.4}$$

(For simplicity, we suppose that this is always feasible, i.e., $\hat{b}_1(t) \neq 0$). This corresponds to a deadbeat control.

4. The Analysis. We will first develop some properties of the parameter estimator, without invoking the form of the control applied.

Define

$$s(t) := \int_\Theta q(t,\theta)\mathrm{d}\,\theta.$$

This is the normalizing factor in (3.3). Note that

$$s(0) \equiv 1$$

from (3.1), since Θ has unit volume.

Lemma 1.

$$\sum_{t=1}^{T} \frac{|y(t) - \phi^T(t-1)\hat{\theta}(t-1)|}{n(t-1)} \leq -\frac{1}{\mu} \ln s(T).$$

Proof. From (3.2),

$$q(t,\theta) \leq q(t-1,\theta) \left[1 - \mu \frac{|y(t) - \phi^T(t-1)\theta|}{n(t-1)} \right].$$

Hence

$$\begin{aligned}
s(t) &= \int_{\Theta} q(t,\theta) \mathrm{d}\,\theta \\
&\leq \int_{\Theta} \left[1 - \mu \frac{|y(t) - \phi^T(t-1)\theta|}{n(t-1)} \right] q(t-1,\theta) \mathrm{d}\,\theta \\
&= s(t-1) - \int_{\Theta} \mu \frac{|y(t) - \phi^T(t-1)\theta|}{n(t-1)} q(t-1,\theta) \mathrm{d}\,\theta \\
&\leq s(t-1) - \left| \int_{\Theta} \frac{\mu(y(t) - \phi^T(t-1))}{n(t-1)} q(t-1,\theta) \mathrm{d}\,\theta \right| \\
&= s(t-1) - \left| \mu \frac{y(t)}{n(t-1)} s(t-1) - \mu \frac{\phi^T(t-1)\hat{\theta}(t-1)}{n(t-1)} s(t-1) \right| \text{(from (3.3))} \\
&= s(t-1) - \mu \frac{s(t-1)}{n(t-1)} \left| y(t) - \phi^T(t-1)\hat{\theta}(t-1) \right| \\
&= \left[1 - \mu \frac{|y(t) - \phi^T(t-1)\hat{\theta}(t-1)|}{n(t-1)} \right] s(t-1).
\end{aligned}$$

So,

$$s(T) \leq s(0) \prod_{t=1}^{T} \left[1 - \mu \frac{|y(t) - \phi^T(t-1)\hat{\theta}(t-1)|}{n(t-1)} \right].$$

Taking logarithms, we obtain

$$\ln \frac{s(0)}{s(T)} \geq \sum_{t=1}^{T} -\ln \left[1 - \mu \frac{|y(t) - \phi^T(t-1)\hat{\theta}(t-1)|}{n(t-1)} \right].$$

Noting $s(0) = 1$, and using $\ln(1-x) \leq -x$, yields the desired result. □

Lemma 2. *Let $S(r,\theta^*)$ denote a small closed sphere of radius r centered at θ^*. Assume $S(r,\theta^*) \subseteq \Theta$. Then, for some constant c,*

$$\sum_{t=1}^{T} \frac{|y(t) - \phi^T(t-1)\hat{\theta}(t-1)|}{n(t-1)}$$

$$\leq \frac{1}{\mu}\left[-\ln(cr^{2p}) + r\ln\left(\frac{1}{1-\mu}\right)\sum_{t=1}^{T}\frac{\|\phi(t-1)\|}{n(t-1)}\right.$$
$$\left. + \ln\left(\frac{1}{1-\mu}\right)\sum_{t=1}^{T}\frac{|y(t)-\phi^T(t-1)\theta^*|}{n(t-1)}\right].$$

Proof. Clearly,

$$\max_{\theta\in S(r,\theta^*)}\sum_{t=1}^{T}\frac{|y(t)-\phi^T(t-1)\theta|}{n(t-1)} \leq \sum_{t=1}^{T}\frac{|y(t)-\phi^T(t-1)\theta^*|}{n(t-1)} + r\sum_{t=1}^{T}\frac{\|\phi(t-1)\|}{n(t-1)}.$$

Now,

$$\begin{aligned}
s(T) &= \int_{\Theta} q(T,\theta)\mathrm{d}\,\theta \\
&\geq \int_{S(r,\theta^*)} q(T,\theta)\mathrm{d}\,\theta \\
&\geq \int_{S(r,\theta^*)} q(0,\theta)(1-\mu)^{\sum_{t=1}^{T}\frac{|y(t)-\phi^T(t-1)\theta|}{n(t-1)}}\mathrm{d}\,\theta \text{ (from (3.2))} \\
&\geq cr^{2p}(1-\mu)^{[\sum_{t=1}^{T}\frac{|y(t)-\phi^T(t-1)\theta^*|}{n(t-1)} + r\sum_{t=1}^{T}\frac{\|\phi(t-1)\|}{n(t-1)}]}.
\end{aligned}$$

Above, cr^{2p} is the volume of $S(r,\theta^*)$. Taking logarithms and using Lemma 1 yields the result. □

Clearly, $\theta^\circ \in \arg\min_{\Theta}\sum_{t=1}^{T}\frac{|y(t)-\phi^T(t-1)\theta|}{n(t-1)}$; in fact $\sum_{t=1}^{T}\frac{|y(t)-\phi^T(t-1)\theta^\circ|}{n(t-1)} = 0$. Hence, by choosing $\theta^* = \theta^\circ$, for small enough r,

$$\begin{aligned}
&\sum_{t=1}^{T}\frac{|y(t)-\phi^T(t-1)\hat{\theta}(t-1)|}{n(t-1)} \\
&\leq \frac{1}{\mu}\left[-\ln(cr^{2p}) + r\ln\left(\frac{1}{1-\mu}\right)\sum_{t=1}^{T}\frac{\|\phi(t-1)\|}{n(t-1)}\right] \qquad (4.1) \\
&= \frac{1}{\mu}\left[-\ln c - 2p\ln r + r\ln\left(\frac{1}{1-\mu}\right)\sum_{t=1}^{T}\frac{\|\phi(t-1)\|}{n(t-1)}\right].
\end{aligned}$$

Now note that for $r = \frac{2p}{x}$,

$$-2p\ln r + rx = 2p + 2p\ln\left(\frac{x}{2p}\right).$$

Hence, with $x = \ln\left(\frac{1}{1-\mu}\right)\sum_{t=1}^{T}\frac{\|\phi(t-1)\|}{n(t-1)}$, we obtain from (4.1),

$$\sum_{t=1}^{T}\frac{|y(t)-\phi^T(t-1)\hat{\theta}(t-1)|}{n(t-1)} \leq c_1 + c_2\ln\sum_{t=1}^{T}\frac{\|\phi(t-1)\|}{n(t-1)}. \qquad (4.2)$$

(Above, we are only treating the case $\sum_{t=1}^{\infty} \frac{\|\phi(t-1)\|}{n(t-1)} = +\infty$, for otherwise $\phi(t) \to 0$, and we are done).

From (4.2), by using the strict minimum phase property of the system, and the Key Technical Lemma from Goodwin and Sin [], it is easy to conclude that the adaptive controller gives $y(t) \to 0$, while keeping signals bounded.

5. Concluding Remarks. We have provided a somewhat novel method of analysis, for an adaptive controller that is not too different from traditional adaptive controllers. This method of analysis may be of interest to others.

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