Stochastic Adaptive Prediction and Model Reference Control *

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Abstract

In a major breakthrough, Guo and Chen [1] have recently shown how to establish the self–optimality and mean square stability of a self–tuning regulator. The ideas there allow us to proceed with the development of a more comprehensive theory of stochastic adaptive filtering, control and identification. In adaptive filtering, we examine both indirect and non–interlaced direct schemes for prediction, using both least–squares and gradient parameter estimation algorithms. In addition to analyzing similar direct adaptive control algorithms, we propose new generalized certainty equivalence adaptive model reference control laws with simultaneous disturbance rejection. We also establish that the parameters converge to the null space of a certain matrix. From this one may deduce the convergence of several adaptive controllers.

1 Introduction

The chief advantage of the stochastic formulation of adaptive problems is its ability to exploit the averaging properties of the disturbance. This allows it to provide better performance in terms of disturbance rejection. The mathematical foundation of this field was laid by Goodwin, Ramadge and Caines [2] and Solo [3], and the various ramifications were explored in Goodwin and Sin [4]. Also, for identification, Lai and Wei [5, 6], and Chen and Guo [7], have determined sharp estimates of the rate of convergence of several parameter estimation algorithms.

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Recently, there have been two further developments which have set the stage for advances in stochastic adaptive filtering and control. In [8], it was shown that much of the asymptotic behavior of such systems, including properties such as self-optimality, self-tuning and strong consistency, could be established in a fairly straightforward manner if only one could prove convergence of the parameter estimates. The establishment of parameter convergence is however non–trivial in general. An exception is in the case of ARX models with Gaussian white noise, where such convergence is easily established for almost every true parameter vector, by resorting to a simple Bayesian embedding procedure [9, 10, 11, 8]. However, it has recently been shown in [12] that the result does not extend to all possible true parameter vectors, implying that nothing can be said for any particular system.

In a major breakthrough, Guo and Chen [1] have shown how to avoid the need to establish parameter convergence. This has long been regarded as a central issue in adaptive control. Without establishing good behavior of the parameter estimates, it is difficult to establish stability of the overall adaptive system. Conversely, without establishing stability, it is difficult to establish good behavior of the parameter estimator. It was for this reason that the stability of the Åström and Wittenmark self–tuning regulator [13] was an open issue ever since its introduction in 1973. By a careful analysis of growth rates, Guo and Chen have been able to resolve some cases of the self–tuning regulator. More recently, Guo [14] has further sharpened the convergence rate of adaptive control, especially for a special case involving adaptive minimum variance control of systems with unit delay and white noise.

By capitalizing on the two developments in [8, 1] it is possible to take a fresh view of stochastic adaptive systems, suggest some new designs, provide clean lines of analysis, and establish many new results. This is the purpose of this paper. We note that part of the results were presented in the Grainger Lectures [15].

We will be concerned throughout with ARMAX systems of the form,

$$A(q^{-1})y(t) = q^{-d}B(q^{-1})u(t) + C(q^{-1})w(t), \tag{1}$$

where u is the input, and y is the output. One can take either a stochastic or a deterministic view with respect to the disturbance w. Some readers may prefer the latter.

The stochastic view is standard. We assume $\{w(t), \mathcal{F}_t\}$ is a martingale difference sequence with some moment conditions, i.e.,

(A.i) $E(w(t)|\mathcal{F}_{t-1}) = 0, \quad$ a.s. $\forall t \geq 0$,

(A.ii) $E(w^2(t)|\mathcal{F}_{t-1}) = \sigma^2 > 0 \quad$ a.s. $\forall t \geq 0$,

(A.iii) $\sup_{t \geq 0} E(|w(t)|^\alpha|\mathcal{F}_{t-1}) < +\infty \quad$ a.s. for some $\alpha > 2$.

For an alternative strictly deterministic view, suppose only that $w(t)$ is a disturbance sequence such that,

(A.i′) $1/N \sum_{t=1}^N (w^2(t) - \sigma^2) = O(N^{-\delta})$ for some $0 < \delta < 2 - \alpha/\alpha$.

Also, for any (measurable) function $f_t(w(0), w(1), \ldots, w(t))$, of $(w(0), \ldots, w(t))$, denoted f_t for short, we require that,

(A.ii′) $\sum_{t=1}^N f_{t-1} w(t) = o\left([\sum_{t=1}^N f_{t-1}^2]^{\frac{1}{2}+\epsilon}\right) + O(1), \forall \epsilon > 0$,

(A.iii′) $\sum_{t=1}^\infty f_{t-1} w(t)$ converges if $\sum_{t=1}^\infty f_{t-1}^2 < +\infty$,

(A.iv′) $\sum_{t=1}^N |f_{t-1}| w^2(t) = O\left(\sum_{t=1}^N |f_{t-1}|\right)$ if $\sup_t |f_t| < +\infty$,

(A.v′) $\lim_{N\to\infty} 1/N \sum_{t=1}^N f_{t-1}^2 = 0$ if $\lim_{N\to\infty} 1/N \sum_{t=1}^N f_{t-1}^2 w^2(t) = 0$ and $\sup_t |f_{t-1}| < +\infty$.

What disturbance sequences satisfy (A.i′-v′)? If $\{w(t), \mathcal{F}_t\}$ satisfies (A.i-iii), i.e., is "white noise," then properties (A.i′-v′) hold with probability one on the events indicated; see [16, 5]. Let us briefly motivate the reasons for these properties. Since f_{t-1} cannot anticipate $w(t)$, in forming the sum $\sum_{t=1}^N f_{t-1} w(t)$, there will be a lot of cancellation of positive and negative terms. The net result is that the term $\sum_{t=1}^N f_{t-1} w(t)$ in (A.ii′) is small in comparison with $\sum_{t=1}^{N} f^2(t)$. Similarly, property (A.iii$'$) also is related to the unpredictability of white noise. Property (A.iv$'$) is due to the bounded average value of $w^2(t)$. Finally property (A.v$'$) is due to the wide band excitation inherent in white noise. If $\sum_{t=1}^{N} f_{t-1}^2(t) w^2(t)$ is small, that can only be the fault of $\sum_{t=1}^{N} f_{t-1}^2$.

We shall focus on three problems vis–a–vis the system (1). In all cases, the coefficients $(a_1, \ldots, a_p, b_0, \ldots, b_h, c_1, \ldots, c_r)$ of the polynomials $A(q^{-1}) = 1 + \sum_{i=1}^{p} a_i q^{-i}$, $B(q^{-1}) = \sum_{i=0}^{h} b_i q^{-i}$, $C(q^{-1}) = 1 + \sum_{i=1}^{r} c_i q^{-i}$ are unknown. However, the delay d is known, and p, h, and r are assumed to be greater than or equal to the true orders of the systems. In all cases, while $u(t)$ and $y(t)$ are measured on–line, $w(t)$ cannot be measured. In the *identification* problem, the goal is to determine the parameters $(a_1, \ldots, a_p, b_0, \ldots, b_h, c_1, \ldots, c_r)$ of (1), or those of an equivalent model. In the *prediction* problem, the goal is to predict $y(t)$ based on past measurements. In the *adaptive feedback control* problem, the goal is to choose the input u so that both y and u behave appropriately. The specification of what constitutes appropriate behavior is a design choice.

In Section 2, we analyze two identification algorithms, each for two possible parametrizations of the system. We also provide the technical framework for many of the ensuing results. In Section 3, we study indirect and direct adaptive predictors, each based on either a least–squares or a gradient algorithm. In Section 4, we propose a generalized certainty equivalence procedure for the design of adaptive control laws. We analyze the performance of adaptive controllers providing pole–zero placement for the servo–response, as well as simultaneous disturbance rejection with respect to the noise response. In Section 5, we study direct adaptive control schemes. In Section 6, we provide a new and general method for deducing adaptive controller parameter convergence. We show that the parameter estimates converge to the null space of a certain covariance matrix. This allows us to easily establish the convergence of several particular adaptive controllers. It provides an easy path for proceeding from self–optimality to self–tuning.

2 Identification

Consider the model,

$$y(t) = x^T(t-s)\beta + d_1 v(t-s) + \cdots + d_\ell v(t-s-\ell+1) + v(t), \tag{2}$$

where $v(t) = \sum_{i=0}^{s-1} f_i w(t-i)$, $x(t)$ is a vector measured on–line, and $\theta^0 = (\beta^T, d_1, \ldots, d_\ell)^T$ is the vector of unknown parameters. Denote $D(q^{-1}) := \sum_{i=0}^{\ell-1} d_{i+1} q^{-i}$ and $F(q^{-1}) = \sum_{i=0}^{s-1} f_i q^{-i}$.

It is clear that, by choosing $s = 1$, (2) includes (1) as a special case. However, by choosing $s = d$, (2) also includes the following reparametrization of (1),

$$y(t) = R(q^{-1})u(t-d) + S(q^{-1})y(t-d) + q^{-d}D(q^{-1})v(t) + v(t). \tag{3}$$

Such an alternative model is useful for prediction, see [17], and also in direct adaptive control. Here $R(q^{-1})$, $S(q^{-1})$, $D(q^{-1})$ and $F(q^{-1})$ are given through the solutions of the equations, $A(q^{-1})F(q^{-1}) + q^{-d}G(q^{-1}) = C(q^{-1})$, $\deg F(q^{-1}) \leq d-1$; $C(q^{-1})\overline{F}(q^{-1}) + q^{-d}\overline{G}(q^{-1}) = 1$, $\deg \overline{F}(q^{-1}) \leq d-1$, $R(q^{-1}) = B(q^{-1})F(q^{-1})\overline{F}(q^{-1})$, $D(q^{-1}) = -\overline{G}(q^{-1})$, and $S(q^{-1}) = G(q^{-1})\overline{F}(q^{-1}) + \overline{G}(q^{-1})$.

The parametrization (3) may also have the advantage of weakening the strict positive real condition for convergence; see [18].

For the identification problem, we consider the following two algorithms to recursively estimate θ^0. The first is a natural generalization of the extended least squares algorithm of [6] to the model (2).

$$\widehat{\theta}(t) = \widehat{\theta}(t-1) + P(t-s)\phi(t-s)[y(t) - \phi^T(t-s)\widehat{\theta}(t-1)], \tag{4}$$

$$P^{-1}(t-s) = P^{-1}(t-s-1) + \phi(t-s)\phi^T(t-s), P(-s) = I, \tag{5}$$

where $\phi^T(t-s-1) := [x^T(t-s-1), \widehat{v}(t-s-1), \ldots, \widehat{v}(t-s-l)]$, and $\widehat{v}(t) := y(t) - \phi^T(t-s)\widehat{\theta}(t)$.

At high sampling rates, as in some acoustic control problems or in signal processing, the following extended stochastic gradient algorithm is useful since it imposes a smaller computational and memory burden.

$$\widehat{\theta}(t) = \widehat{\theta}(t-1) + \frac{\phi(t-s)}{r(t-s)}[y(t) - \phi^T(t-s)\widehat{\theta}(t-1)], \tag{6}$$

$$r(t-s) = r(t-s-1) + \|\phi(t-s)\|^2; \quad r(-s) = 1. \tag{7}$$

Theorems 2.1 and 2.2 simultaneously cover both the models (1) and (3). For the model (1) they are the same as in [5, 6] and [7], and hence not new. However, for the reparametrized model (3), which is particularly useful in direct adaptive prediction and control, the results are new. Traditionally, for such models with delay d greater than 1, interlaced algorithms have been analyzed in the literature; see [17, 19, 20]. However, that has only been because of the inability to analyze the non–interlaced form (4,5) or (6,7). Interlacing requires extra storage, is cumbersome to implement, and is anyway not used in practice. Recently, Lai and Ying [21] and Radenkovic [22] have proved the convergence of a non–interlaced adaptive predictor based on the stochastic gradient algorithm. However, the SPR condition required by them is stronger than that needed for the interlaced algorithm. This leaves open the possibility that interlacing may perhaps be advantageous. Moreover, the techniques used in [21, 22] do not appear to be generalizable to the desirable extended least squares algorithm.

Theorem 2.1. *Consider the extended least squares parameter estimator (4,5) applied to the model (2,A.i-A.iii). Assume that,*[1]

$$\mathrm{Re}\left[(1 + e^{j\omega s}D(e^{j\omega}))^{-1} - 1/2\right] > 0 \text{ for } 0 \le \omega \le 2\pi. \tag{8}$$

Then

(i)

$$\|\widehat{\theta}(N) - \theta^0\|^2 = O\left(\frac{\log \lambda_{\max} R(N-s)}{\lambda_{\min} R(N-s)}\right) \qquad a.s. \tag{9}$$

[1]From the Nyquist stability criterion, it is easy to see that the strict positive real conditions (8) or (27) imply the stability of $1 + q^{-d}D(q^{-1})$, i.e., that all the roots of $1 + z^d D(z)$ are outside the closed unit disk.

ii)

$$\sum_{t=1}^{N} (\widehat{v}(t) - v(t))^2 = O(\log \lambda_{\max} R(N-s) \text{ a.s.} \tag{10}$$

iii)

$$\sum_{t=1}^{N} \frac{[y(t) - \phi^{\mathrm{T}}(t-s)\widehat{\theta}(t-k) - v(t)]^2}{1 + \phi^{\mathrm{T}}(t-s)P(t-s-k)\phi(t-s)} = O(\log \lambda_{\max} R(N-s)) \text{ a.s.}, \forall k < \infty. \tag{11}$$

(iv) Define the "optimal regressor,"

$$\phi_0(t) := [x^T(t), v(t), \ldots, v(t-l+1)]^T,$$

and the related quantities,

$$\begin{aligned} R_0(N) &:= I + \sum_{t=1}^{N} \phi_0(t)\phi_0^T(t), \\ r_0(N) &:= \text{trace } R_0(N). \end{aligned}$$

If

$$r_0(N) \to +\infty \quad \text{and} \quad \frac{\log \lambda_{\max}(R_0(N))}{\lambda_{\min}(R_0(N))} \to 0 \text{ a.s.}, \tag{12}$$

then

$$\widehat{\theta}(N) \to \theta^0 \quad \text{a.s.}$$

Proof. Define $\widetilde{\theta}(t) := \widehat{\theta}(t) - \theta^0$ and $V(t) := \widetilde{\theta}^T(t)P(t-s)\widetilde{\theta}(t)$. Proceeding in the standard way,

$$\begin{aligned} V(t-1) = V(t) - 2\phi^T(t-s)\widetilde{\theta}(t)\left[1/2\phi^T(t-s)\widetilde{\theta}(t) + (\widehat{v}(t) - v(t))\right] \\ + \phi^T(t-s)P(t-s-1)\phi(t-s)\widehat{v}^2(t) - 2\phi^T(t-s)\widetilde{\theta}(t)v(t). \end{aligned} \tag{13}$$

We will now show that the last term above satisfies,

$$\sum_{t=1}^{N} \phi^T(t-s)\widetilde{\theta}(t)v(t) = o\left(\sum_{t=1}^{N}\left(\phi^T(t-s)\widetilde{\theta}(t)\right)^2\right) + O(\log r(N-s)), \tag{14}$$

where $r(t) :=$ trace $P^{-1}(t)$. Above, and throughout, all equations hold at least almost surely. Define the operator,

$$\mathcal{H}_t(z) = z - E[z|\mathcal{F}_{t-s}]. \tag{15}$$

We have

$$\sum_{t=1}^{N} \phi^T(t-s)\widetilde{\theta}(t)v(t) = \sum_{t=1}^{N} \phi^T(t-s)\mathcal{H}_t(\widehat{\theta}(t))v(t) + \sum_{t=1}^{N} \phi^T(t-s)E[\widetilde{\theta}(t)|\mathcal{F}_{t-s}]v(t). \tag{16}$$

The first term on the right hand side above is bounded as follows. From (4), for $i = 0, \ldots, s-2$,

$$P^{-1}(t-s-i)\mathcal{H}_t(\widehat{\theta}(t-i)) = \phi(t-s-i)\mathcal{H}_t(v(t-i)) + P^{-1}(t-s-i-1)\mathcal{H}_t(\widehat{\theta}(t-i-1)), \tag{17}$$

$$\mathcal{H}_t(\widehat{\theta}(t-s+1)) = P(t-2s+1)\phi(t-2s+1)\mathcal{H}_t(v(t-s+1)). \tag{18}$$

So

$$\mathcal{H}_t(\widehat{\theta}(t)) = P(t-s)\sum_{i=0}^{s-1} \phi(t-s-i)\mathcal{H}_t(v(t-i)). \tag{19}$$

Hence, using (A-ii$'$) and (A-iv$'$) we have

$$\begin{aligned} &\textstyle\sum_{t=1}^N \phi^T(t-s)\mathcal{H}_t(\widehat{\theta}(t))v(t) = \sum_{t=1}^N \phi^T(t-s)P(t-s)\sum_{i=0}^{s-1}\phi(t-s-i)\mathcal{H}_t(v(t-i))v(t) \\ &= O\left(\textstyle\sum_{i=0}^{s-1}(\sum_{t=1}^N \phi^T(t-s)P(t-s)\phi(t-s))^{1/2}(\sum_{t=1}^N \phi^T(t-s-i)P(t-s)\phi(t-s-i))^{1/2}\right) \\ &\qquad + O(1) = O(\log r(N-s)). \end{aligned} \tag{20}$$

For the second term on the right hand side of (16), using (A-iv$'$) we have,

$$\begin{aligned} &\textstyle\sum_{t=1}^N \phi^T(t-s)E[\widetilde{\theta}(t)|\mathcal{F}_{t-s}]v(t) = o\left(\sum_{t=1}^N \left(\phi^T(t-s)E[\widetilde{\theta}(t)|\mathcal{F}_{t-s}]\right)^2\right) + O(1) \\ &= o\left(\textstyle\sum_{t=1}^N \left(\phi^T(t-s)\widetilde{\theta}(t)\right)^2\right) + o\left(\sum_{t=1}^N(\phi^T(t-s)\mathcal{H}_t\left(\widehat{\theta}(t)\right)^2\right) + O(1) \text{ (using (15))}. \end{aligned} \tag{21}$$

By using (19), the second term on the RHS above can be bounded as in (20) to give,

$$\begin{aligned} &\textstyle\sum_{t=1}^N(\phi^{\mathrm{T}}(t-s)\mathcal{H}_t(\widehat{\theta}(t)))^2 = O\left(\sum_{i=1}^{s-1}\sum_{t=1}^N(\phi^{\mathrm{T}}(t-s)P(t-s)\phi(t-s-i))^2(\mathcal{H}_t(v(t-i)))^2\right) \\ &= O\left(\textstyle\sum_{i=1}^{s-1}\sum_{t=1}^N \phi^{\mathrm{T}}(t-s)P(t-s)\phi(t-s)\phi^{\mathrm{T}}(t-s-i)P(t-s)\phi(t-s-i)\right) \\ &= O(\log r(N-s)). \end{aligned} \tag{22}$$

From (16,20,21,22), we thus establish (14).

Next, as is usual, it can be shown that,

$$(1+q^{-s}D(q^{-1}))(\widehat{v}(t)-v(t)) = -\phi^T(t-s)\widetilde{\theta}(t). \tag{23}$$

From the SPR condition (8), the second term on the right hand side of (13) satisfies,

$$\begin{aligned}&\textstyle\sum_{t=1}^{N} -\phi^T(t-s)\widetilde{\theta}(t)\left[1/2\phi^T(t-s)\widetilde{\theta}(t)+\widehat{v}(t)-v(t)\right]\\ &\geq \epsilon_1 \textstyle\sum_{t=1}^{N}\left[\phi^T(t-s)\widetilde{\theta}(t)\right]^2+\epsilon_2\sum_{t=1}^{N}(\widehat{v}(t)-v(t))^2+O(1),\end{aligned} \tag{24}$$

for some ϵ_1, $\epsilon_2 > 0$.

Substituting (24) and (14) in (13), we get (9) and (10). Also, from (10), we obtain,

$$r(t) \sim r_0(t) \text{ a.s.} \tag{25}$$

Additionally, for any k, we have

$$\begin{aligned}&\sum_{t=1}^{N}\frac{(y(t)-\phi^T(t-s)\widehat{\theta}(t-k)-v(t))^2}{1+\phi^T(t-s)P(t-s-k)\phi(t-s)}\\ &\leq 2\sum_{t=1}^{N}(\widehat{v}(t)-v(t))^2+2\sum_{t=1}^{N}\frac{[\phi^T(t-s)(\widehat{\theta}(t)-\widehat{\theta}(t-k))]^2}{1+\phi^T(t-s)P(t-s-k)\phi(t-s)}\\ &\leq O(\log r(N-s))+2\sum_{t=1}^{N}\frac{\left[\sum_{i=0}^{k-1}\phi^T(t-s)P(t-s-i-1)\phi(t-s-i)\widehat{v}(t-i)\right]^2}{1+\phi^T(t-s)P(t-s-k)\phi(t-s)}\\ &= O(\log r(N-s))+O\left(\textstyle\sum_{i=1}^{k-1}\sum_{t=1}^{N}\phi^T(t-s-i)P(t-s-i-1)\phi(t-s-i)\widehat{v}^2(t-i)\right)\\ &= O(\log r(N-s)),\end{aligned}$$

which yields (11). Similarly, one can show

$$\sum_{t=1}^{N}[\frac{\phi^{\mathrm{T}}(t-s)\widetilde{\theta}(t-k)]^2}{1+\phi^{\mathrm{T}}(t-s)P(t-s-k)\phi(t-s)} = O(\log\lambda_{\max}R(N-s)) = O(\log r(N-s)). \tag{26}$$

The proof of parameter consistency, when (12) holds, can now be completed as in [5, 6]. $\square$

Theorem 2.2. *Consider the extended stochastic gradient parameter estimator (6,7) applied to the model (2,A.i-iii). Assume*

$$\mathrm{Re}\left[1 + e^{j\omega s} D(e^{j\omega})\right] > 0 \text{ for } 0 \le \omega \le 2\pi. \tag{27}$$

Then

(i)

$$\sum_{t=1}^{+\infty} \frac{(\widehat{v}(t) - v(t))^2}{r(t-s-1)} < \infty, \text{ a.s.} \tag{28}$$

(ii)

$$\sum_{t=1}^{\infty} \|\widehat{\theta}(t) - \widehat{\theta}(t-k)\|^2 < \infty, \text{ a.s.}, \forall k < \infty. \tag{29}$$

(iii)

$$\|\widetilde{\theta}(t)\| \text{ converges a.s.} \tag{30}$$

(iv) If

$$r_0(N) \to +\infty \text{ and } \sup_N \left(\frac{\lambda_{\max}(R_0(N))}{\lambda_{\min}(R_0(N))}\right) < +\infty \text{ a.s.}, \tag{31}$$

then

$$\widehat{\theta}(N) \to \theta^0 \qquad \text{a.s.}$$

Proof. Defining $V(t) := \|\widetilde{\theta}(t)\|^2$, we obtain the Lyapunov recursion,

$$\begin{aligned} V(t) - \frac{2\phi^T(t-s)\widetilde{\theta}(t)(\widehat{v}(t) - v(t))}{r(t-s-1)} - \frac{2\phi^T(t-s)E[\widetilde{\theta}(t)|\mathcal{F}_{t-s}]v(t)}{r(t-s-1)} + \frac{\|\phi(t-s)\|^2 \widehat{v}^2(t)}{r^2(t-s-1)} \\ = V(t-1) + \frac{2\phi^T(t-s)\mathcal{H}_t(\widehat{\theta}(t))v(t)}{r(t-s-1)}. \end{aligned} \tag{32}$$

From (6), noting that $y(t) - v(t)$ is $\mathcal{F}_{t-s}$-measurable, we get,

$$\begin{aligned} \mathcal{H}_t(\widehat{\theta}(t-i)) &= \frac{r(t-s-i)I - \phi(t-s-i)\phi^{\mathrm{T}}(t-s-i)}{r(t-s-i)} \mathcal{H}_t(\widehat{\theta}(t-i-1)) \\ &+ \frac{\phi(t-s-i)}{r(t-s-i)} \mathcal{H}_t(v(t-i)), \end{aligned} \tag{33}$$

and

$$\mathcal{H}_t(\widehat{\theta}(t-s+1)) = \frac{\phi(t-2s+1)}{r(t-2s+1)}\mathcal{H}_t(v(t-s+1))\,. \tag{34}$$

Using (33) with $i=0$, we obtain,

$$\begin{aligned}
\sum_{t=1}^{N}\left|\frac{\phi^T(t-s)\mathcal{H}_t(\widehat{\theta}(t))v(t)}{r(t-s-1)}\right| &= \sum_{t=1}^{N}\left|\frac{\phi^T(t-s)\mathcal{H}_t(\widehat{\theta}(t-1))v(t)}{r(t-s)} + \frac{\|\phi(t-s)\|^2}{r(t-s)r(t-s-1)}v^2(t)\right| \\
&\leq \sum_{t=1}^{N}\left(\frac{\|\phi(t-s)\|^2 v^2(t)}{r^2(t-s)}\right)^{1/2}\left(\sum_{t=1}^{N}\|\mathcal{H}_t(\widehat{\theta}(t-1))\|^2\right)^{1/2} + \sum_{t=1}^{N}\frac{\|\phi(t-s)\|^2}{r(t-s)r(t-s-1)}v^2(t) \\
&= O\left(\left(\textstyle\sum_{t=1}^{N}\|\mathcal{H}_t(\widehat{\theta}(t-1))\|^2\right)^{1/2}\right) + O(1).
\end{aligned} \tag{35}$$

Above, we have used the fact that $\dfrac{\|\phi(t-s)\|^2}{r(t-s)r(t-s-1)}$ is summable and (A.iv′). To show that the first term on the RHS above is also summable, note that from (33), for $i=1,\ldots,s-2$, $\|\mathcal{H}_t(\widehat{\theta}(t-i))\| \leq \|\mathcal{H}_t(\widehat{\theta}(t-i-1))\| + \frac{\|\phi(t-s-i)\|}{r(t-s-i)}\,|\mathcal{H}_t(v(t-i))|$, and from (34), $\|\mathcal{H}_t(\widehat{\theta}(t-s+1))\| \leq \frac{\|\phi(t-2s+1)\|}{r(t-2s+1)}\,|\mathcal{H}_t(v(t-s+1))|$. Hence, $\|\mathcal{H}_t(\widehat{\theta}(t-1))\| \leq \sum_{i=1}^{s-1}\frac{\|\phi(t-s-i)\|}{r(t-s-i)}\,|\mathcal{H}_t(v(t-s-i))|$, and it follows that the first summation on the RHS of (35) is summable. So the last term in (32) is summable.

Turning attention to the third term on the left hand side of (32), note first that from (35),

$$\begin{aligned}
\sum_{t=1}^{N}\frac{\phi^T(t-s)E[\widetilde{\theta}(t)|\mathcal{F}_{t-s}]v(t)}{r(t-s-1)} &= o\left(\sum_{t=1}^{N}\left(\frac{\phi^T(t-s)E[\widetilde{\theta}(t)|\mathcal{F}_{t-s}]}{r(t-s-1)}\right)^2\right) + O(1) \\
&= o\left(\sum_{t=1}^{N}\left(\frac{\phi^T(t-s)\widetilde{\theta}(t))}{r(t-s-1)}\right)^2\right) + o\left(\sum_{t=1}^{N}\left(\frac{\phi^T(t-s)\mathcal{H}_t(\widehat{\theta}(t))}{r(t-s-1)}\right)^2\right) + O(1) \\
&= o\left(\sum_{t=1}^{N}\left(\frac{\phi^T(t-s)\widetilde{\theta}(t))}{r(t-s-1)}\right)^2\right) + O\left(\sum_{t=1}^{N}\left(\frac{\phi^{\mathrm{T}}(t-s)\mathcal{H}_t(\widehat{\theta}(t-1))}{r(t-s)}\right)^2\right) \\
&\quad + O\left(\sum_{t=1}^{n}\left(\frac{\|\phi(t-s)\|^2}{r(t-s)r(t-s-1)}\right)^2 v^2(t)\right) \quad \text{(from (33))} \\
&= o\left(\sum_{t=1}^{N}\left(\frac{\phi^T(t-s)\widetilde{\theta}(t))}{r(t-s-1)}\right)^2\right) + O(1).
\end{aligned}$$

From (23) and the SPR assumption (27), we have (see [3, 4])

$$S(N) = \sum_{t=1}^{N}[-\phi^{\mathrm{T}}(t-s)\widetilde{\theta}(t)\,(\widehat{v}(t)-v(t)) - \epsilon(\widehat{v}(t)-v(t))^2] + S(0) \geq 0\,, \tag{36}$$

for some constant $\epsilon > 0$, and random variable $S(0) < \infty$ a.s. Summing by parts, we have

$$\sum_{t=1}^{N} \frac{S(t)-S(t-1)}{r(t-s-1)} = \frac{S(N)}{r(N-s-1)} + \sum_{t=1}^{N-1} S(t)\left(\frac{1}{r(t-s-1)} - \frac{1}{r(t-s)}\right) - \frac{S(0)}{r(-s)}. \quad (37)$$

Since $r(t)$ is monotonically nondecreasing, it follows from (36), (37) that

$$\sum_{t=1}^{N} \frac{[-\phi^{\mathrm{T}}(t-s)\widetilde{\theta}(t)\,(\widehat{v}(t)-v(t)) - \epsilon(\widehat{v}(t)-v(t))^2]}{r(t-s-1)} \geq -\frac{S(0)}{r(-s)}. \quad (38)$$

Using (38,23,36), and the summability of (35), in (32), we obtain (28) and

$$\sum_{t=1}^{\infty} \frac{\|\phi(t-s)\|^2 \widehat{v}^2(t)}{r^2(t-s-1)} \ < \ \infty,$$

which yields (29). Also, from (28),

$$r(t) \sim r_0(t) \text{ a.s.} \quad (39)$$

For any k, we have

$$\begin{aligned}
&\sum_{t=1}^{\infty} \frac{[y(t) - \phi^T(t-s)\widehat{\theta}(t-k) - v(t)]^2}{r(t-s)} \\
&\leq \sum_{t=1}^{\infty} \frac{2[y(t) - \phi(t-s)\widehat{\theta}(t) - v(t)]^2}{r(t-s)} + \sum_{t=1}^{\infty} \frac{2[\phi^T(t-s)(\widehat{\theta}(t) - \widehat{\theta}(t-k))]^2}{r(t-s)} \\
&\leq \sum_{t=1}^{\infty} \frac{2[\widehat{v}(t) - v(t)]^2}{r(t-s-1)} + \ 2\left(\sum_{t=1}^{\infty} \frac{\|\phi(t-s)\|^2}{r^2(t-s)}\right)^{1/2} \left(\sum_{t=1}^{\infty} \|\widehat{\theta}(t) - \widehat{\theta}(t-k)\|^2\right)^{1/2} < \infty. \quad (40)
\end{aligned}$$

Similarly,

$$\sum_{t=1}^{\infty} \frac{[\phi^T(t-s)\widetilde{\theta}(t-k)]^2}{r(t-s)} < \infty. \quad (41)$$

We now show (30). From (6), we have

$$\begin{aligned}
\|\widetilde{\theta}(t)\|^2 - \|\widetilde{\theta}(t-1)\|^2 &= \tfrac{\|\phi(t-s)\|^2}{r^2(t-s)}[y(t) - \phi^T(t-s)\widehat{\theta}(t-1) - v(t)]^2 \\
&+ \tfrac{2\phi^T(t-s)\widetilde{\theta}(t-1)}{r(t-s)}[y(t) - \phi^T(t-s)\widetilde{\theta}(t-1) - v(t)] + \tfrac{2\|\phi(t-s)\|^2}{r^2(t-s)} v^2(t) \\
&+ \tfrac{2\phi^T(t-s)\widetilde{\theta}(t-1)}{r(t-s)} v(t) + \tfrac{\|\phi(t-s)\|^2}{r^2(t-s)}[y(t) - \phi^T(t-s)\widehat{\theta}(t-1) - v(t)]v(t).
\end{aligned}$$

From (40,23,41) and Schwarz inequality, it is clear that the first three terms on right hand side above are absolutely summable. From (29) and A.iii$'$), the sum of the last two terms also converge. This establishes (30).

Now, multiplying (6) by $R(t-s)$ and assuming $r(t)\to\infty$, we have

$$\frac{1}{r(N-s)}\left\|R(N-s)\widetilde{\theta}(N)\right\| \le \frac{1}{r(N-s)}\left\|\sum_{t=1}^{N}\phi(t-s)\phi(t-s)^T\widetilde{\theta}(t-1)\right\|$$
$$+\frac{1}{r(N-s)}\left\|\sum_{t=1}^{N}\frac{R(t-s)}{r(t-s)}\phi(t-s)(y(t)-\phi^T(t-s)\widehat{\theta}(t-1)-v(t))\right\|$$
$$+\frac{1}{r(N-s)}\left\|\sum_{t=1}^{N}\frac{R(t-s)}{r(t-s)}\phi(t-s)v(t)\right\|+o(1).$$

The terms on the right hand side converge to zero by the Schwarz inequality and (40,41,A.ii′). Hence,

$$\lim_{N\to\infty}\frac{1}{r(N-s)}\|R(N-s)\widetilde{\theta}(N)\| = 0 \quad \text{a.s.} \tag{42}$$

Finally, from (28),

$$\lim_{N\to\infty}\frac{1}{r(N-s)}(R(N-s)-R_0(N-s)) = 0 \quad \text{a.s.} \tag{43}$$

The result (iv) now follows from (31,42,39,43). □

We should note that the condition (31) of (iv) above can be weakened. By a treatment similar to that in Theorem 4.5 of Chen and Guo [23], the convergence of $\hat{\theta}(N)$ to θ^0 can be shown even if the condition number in (31) has a divergence rate of $O(\log^{1/4} r(t))$.

3 Adaptive Prediction

For adaptive prediction, one may adopt either an indirect or a direct approach. For the indirect approach, we begin with

$$E[y(t+d)|\mathcal{F}_t] = B(q^{-1})F'(q^{-1})u(t)+G'(q^{-1})y(t)+M(q^{-1})w(t), \tag{44}$$

where $F'(q^{-1})$, $G'(q^{-1})$ and $M'(q^{-1})$ are polynomials which satisfy $A(q^{-1})F'(q^{-1}) + q^{-d}G'(q^{-1}) = 1, \deg F'(q^{-1}) \le d-1$, $M(q^{-1}) =$ Truncation $(F'(q^{-1})C(q^{-1}), d)$. Here, Truncation $(\cdot)$ is defined by,

$$\text{Truncation }\left(\sum_{i=0}^{n}\alpha_i q^{-i}, d\right) := \sum_{i=0}^{n-d}\alpha_{i+d}q^{-i}.$$

Note also that the error of the conditional mean prediction (44) is given by,

$$y(t+d) - E[y(t+d)|\mathcal{F}_t] = F(q^{-1})w(t), \tag{45}$$

where F, along with another polynomial G, satisfies,

$$A(q^{-1})F(q^{-1}) + q^{-d}G(q^{-1}) = C(q^{-1}), \deg F(q^{-1}) \leq d-1. \tag{46}$$

It is easy to see that $F(q^{-1}) := F'(q^{-1})C(q^{-1}) - q^{-d}M(q^{-1})$.

The indirect adaptive predictor is based on the certainty equivalent approach. It replaces all the unknown quantities in (44), that is $B(q^{-1})$, $F'(q^{-1})$, $G'(q^{-1})$, $M(q^{-1})$ and $w(t)$, by their estimates. The procedure is as follows. First, estimates $\widehat{A}(q^{-1}, t)$, $\widehat{B}(q^{-1}, t)$ and $\widehat{C}(q^{-1}, t)$ are made of the model (1), using either the extended least squares algorithm (4,5), or the extended stochastic gradient algorithm (6,7) with $s = 1$. Estimates are then obtained for $\widehat{F}'(q^{-1}, t) =: \sum_{i=0}^{d-1} \widehat{f}_i'(t)q^{-i}$, $\widehat{G}'(q^{-1}, t)$, and $\widehat{M}(q^{-1}, t)$ by solving the equations,[2]

$$\widehat{A}(q^{-1}, t)\widehat{F}'(q^{-1}, t) + q^{-d}\widehat{G}'(q^{-1}, t) = 1, \deg \widehat{F}'(q^{-1}, t) \leq d-1, \tag{47}$$

$$\widehat{M}(q^{-1}, t) = \text{ Truncation } (\widehat{F}'(q^{-1}, t)\widehat{C}(q^{-1}, t), d).$$

An estimate $\widehat{w}(t)$ for $w(t)$ is automatically generated by the parameter estimation algorithm (4,5) or (6, 7); it is denoted as $\widehat{v}(t)$ there. The adaptive prediction is then taken as,

$$\widehat{y}(t+d|t) := (\widehat{B}(q^{-1}, t)\widehat{F}'(q^{-1}, t))u(t) + \widehat{G}'(q^{-1}, t)y(t) + \widehat{M}(q^{-1}, t)\widehat{w}(t). \tag{48}$$

The direct method of adaptive prediction is also based on the certainty equivalent approach, but it uses the parametrized model (3) rather than (1). From (3) it is clear that

$$E[y(t+d)|\mathcal{F}_t] = R(q^{-1})u(t) + S(q^{-1})y(t) + D(q^{-1})v(t).$$

From either the extended least squares algorithm (4,5) or the extended stochastic gradient algorithm (6,7) with $s = d$, we obtain estimates $\widehat{R}(q^{-1}, t)$, $\widehat{S}(q^{-1}, t)$, $\widehat{D}(q^{-1}, t)$ and $\widehat{v}(t)$. The direct adaptive prediction is then simply taken as,

$$\widehat{y}(t+d|t) = \widehat{R}(q^{-1}, t)u(t) + \widehat{S}(q^{-1}, t)y(t) + \widehat{D}(q^{-1}, t)\widehat{v}(t),$$

[2]When multiplying time varying polynomials such as $\widehat{A}(q^{-1}, t)$ and $\widehat{F}(q^{-1}, t)$ we ignore the time index t, and treat them merely as polynomials in q^{-1}.

i.e.,

$$\widehat{y}(t+d|t) = \phi^T(t)\widehat{\theta}(t). \tag{49}$$

Theorem 3.1. *Consider the indirect adaptive predictor (48) using the extended least squares parameter estimator (4,5) for the model (1,A.i-iii). Assume that*

$$\mathrm{Re}[C^{-1}(e^{j\omega})] > 1/2 \ \textit{for}\ 0 \le \omega \le 2\pi. \tag{50}$$

Suppose

$$y^2(t) + u^2(t) = O\left(\frac{t}{\log^d t}\right) \qquad \text{a.s.} \tag{51}$$

Then the adaptive predictor is self–optimizing in the sense that

$$\lim_{N\to\infty} 1/N \sum_{t=1}^{N} (\widehat{y}(t+d|t) - E(y(t+d)|\mathcal{F}_t))^2 = 0 \qquad \text{a.s.} \tag{52}$$

If $y(t)$, $u(t)$ and $w(t)$ are uniformly bounded, then the accumulated "regret" satisfies,

$$\sum_{t=1}^{N} [\widehat{y}(t+d|t) - E(y(t+d)|\mathcal{F}_t)]^2 = o(\log^{d+1} N) \qquad \text{a.s.} \tag{53}$$

Proof. First note that,

$$\begin{aligned} \widehat{A}(q^{-1},t)y(t+d-j) &= \widehat{B}(q^{-1},t)u(t-j) + \widehat{C}(q^{-1},t)\widehat{w}(t+d-j) \\ &+C(q^{-1})(w(t+d-j) - \widehat{w}(t+d-j)) - \phi^{\mathrm{T}}(t+d-j-1)\widetilde{\theta}(t). \end{aligned} \tag{54}$$

Hence, multiplying by $\widehat{F}'(q^{-1},t)$ we obtain the representation,

$$\begin{aligned} y(t+d) - \widehat{F}(q^{-1},\, t)w(t+d) - \widehat{y}(t+d|t) &= (\widehat{F}'(q^{-1},\, t)C(q^{-1}))(w(t+d) - \widehat{w}(t+d)) \\ &+\widehat{F}(q^{-1},\, t)(\widehat{w}(t+d) - w(t+d)) - \textstyle\sum_{i=0}^{d-1} \widehat{f}_i'(t)\phi^T(t+d-i-1)\widetilde{\theta}(t), \end{aligned} \tag{55}$$

where $\widehat{F}(q^{-1},\, t) := \widehat{F}'(q^{-1},\, t)\widehat{C}(q^{-1},\, t) - q^{-d}\widehat{M}(q^{-1},\, t)$ satisfies

$$\widehat{A}(q^{-1},\, t)\widehat{F}(q^{-1},\, t) + q^{-d}\widehat{G}(q^{-1},\, t) = \widehat{C}(q^{-1},\, t), \deg \widehat{F}(q^{-1},\, t) \le d-1, \tag{56}$$

From (9) and the division algorithm for solving (56, 47),

$$|\widehat{f}_i(t)|^2 + |\widehat{f}_i'(t)|^2 = O(\log^{d^{-1}} r(t)) \qquad \text{a.s.} \tag{57}$$

From (57,10), the first and second terms on the right hand side of (55) satisfy,

$$\sum_{t=1}^{N}\left[\left(\widehat{F}'(q^{-1},\,t)C(q^{-1})-\widehat{F}(q^{-1},\,t)\right)(w(t+d)-\widehat{w}(t+d)\right]^2=O(\log^d r(N))\quad \text{a.s.} \tag{58}$$

The last term on the right hand side of (55) satisfies,

$$\sum_{t=1}^{N}\left(\sum_{i=0}^{d-1}\widehat{f}_i'(t)\phi^T(t+d-i-1)\widetilde{\theta}(t)\right)^2=O((\log^{d-1} r(N))\sum_{i=0}^{d-1}\sum_{t=1}^{N}(\phi^T(t+d-i-1)\widetilde{\theta}(t))^2\quad \text{a.s.} \tag{59}$$

Using the idea of [1], let us define $\alpha(t)=\dfrac{(\phi^T(t-1)\widetilde{\theta}(t-d+i))^2}{1+\phi^T(t-1)P(t-2d+i)\phi(t-1)}$. Then using (25,26,10), and since $\lim_{t\to\infty}(P(t-1)-P(t-k))=0$,

$$\begin{aligned}
&\textstyle\sum_{t=1}^{N}(\phi^T(t-1)\widetilde{\theta}(t-d+i))^2\\
&\le 2\textstyle\sum_{t=1}^{N}\alpha(t)+\sum_{t=1}^{N}\alpha(t)\phi^T(t-1)(P(t-2d+i)-P(t-1))\phi(t-1)\\
&=O(\log r_o(N-1))+o(\log r_0(N-1))\max_{1\le t\le N}\|\phi(t-1)\|^2\\
&=O(\log r_o(N-1))\\
&+o(\log r_0(N-1))[\max_{1\le t\le N}\|\phi(t-1)-\phi_0(t-1)\|^2+\max_{1\le t\le N}\|\phi_0(t-1)\|^2]\\
&=o(\log^2 r_0(N-1))+o(\log r_0(N))\max_{1\le t\le N}\|\phi_0(t-1)\|^2\\
&=\begin{cases} o(N/\log^{d-1}N) & \text{if (51) holds,}\\ o(\log^2 r_0(N)) & \text{if } y(t),u(t),w(t) \text{ are uniformly bounded.}\end{cases}
\end{aligned} \tag{60}$$

Substituting the bounds (58,59,60) in (55) yields,

$$\begin{aligned}
&\textstyle\sum_{t=1}^{N}[y(t+d)-\widehat{F}(q^{-1},\,t)w(t+d)-\widehat{y}(t+d|t)]^2\\
&=\begin{cases} o(N) & \text{if (51) holds,}\\ O(\log^{d+1}N) & \text{if } y(t),u(t),w(t) \text{ are bounded.}\end{cases}
\end{aligned} \tag{61}$$

Now note from (A.ii′) that since $(\widehat{y}(t+d))-E(y(t+d)|\mathcal{F}_t)$ is $\mathcal{F}_t$-measurable,

$$\begin{aligned}
&\textstyle\sum_{t=1}^{N}[y(t+d)-\widehat{F}(q^{-1},\,t)w(t+d)-\widehat{y}(t+d|t)]^2\\
&=\textstyle\sum_{t=1}^{N}[\widehat{y}(t+d|t)-E(y(t+d|\mathcal{F}_t))+(\widehat{F}(q^{-1},\,t)-F(q^{-1}))w(t+d)]^2\\
&=\textstyle\sum_{t=1}^{N}[\widehat{y}(t+d|t)-E(y(t+d)|\mathcal{F}_t)]^2\\
&+\textstyle\sum_{t=1}^{N}[(\widehat{F}(q^{-1},\,t)-F(q^{-1}))w(t+d)]^2\\
&+o(log^{\frac{d}{2}})\left\{\textstyle\sum_{t=1}^{N}[\widehat{y}(t+d|t)-E(y(t+d)|\mathcal{F}_t)]^2\right\}^{\frac{1}{2}+\epsilon},
\end{aligned} \tag{62}$$

where in the last equality we have also used (57,51). The results (52,53) can be shown by contradiction from (61,62) since the second term term on the right hand side above is nonnegative. $\square$

Theorem 3.2. *Consider the indirect adaptive predictor (48), using the extended stochastic gradient parameter estimator (6,7), for the model (1,A.i-iii). Suppose*

$$\mathrm{Re}[C(e^{j\omega})] > 0 \; for \; 0 \leq \omega \leq 2\pi, \tag{63}$$

$$\sup_N 1/N \sum_{t=1}^{N} (y^2(t) + u^2(t)) < +\infty \quad \text{a.s.}, \tag{64}$$

hold. Then the adaptive predictor is self–optimizing in the sense of (52).

Proof. As in Theorem 3.1, we obtain the representation (55). From (30), and the continuous dependence of $\widehat{F}(q^{-1}, t)$ and $\widehat{F}'(q^{-1}, t)$ on $\widehat{\theta}(t)$, it follows that the coefficients $\widehat{f}_i(t)$ and $\widehat{f}_i'(t)$ are uniformly bounded. Suppose $r(t)$ is bounded, then (52) follows trivially. When $r(t) \to +\infty$, using Kronecker's Lemma, and (55,45,28,40,39,64), we have

$$\lim_{N\to\infty} 1/N \sum_{t=1}^{N} [\widehat{y}(t+d|t) - E(y(t+d)|\mathcal{F}_t) + (\widehat{F}(q^{-1}, t) - F(q^{-1}))w(t+d)]^2 = 0 \quad \text{a.s.}$$

Similar to (62),

$$\begin{aligned} &\textstyle\sum_{t=1}^{N} [\widehat{y}(t+d|t) - E(y(t+d|\mathcal{F}_t)) + (\widehat{F}(q^{-1}, t) - F(q^{-1}))w(t+d)]^2 \\ &\quad = (1+o(1)) \textstyle\sum_{t=1}^{N} [\widehat{y}(t+d|t) - E(y(t+d)|\mathcal{F}_t)]^2 \\ &\qquad + \textstyle\sum_{t=1}^{N} [(\widehat{F}(q^{-1}, t) - F(q^{-1}))w(t+d)]^2. \end{aligned} \tag{65}$$

The result then follows. $\square$

Theorem 3.3. *Consider the direct adaptive predictor (49), using the extended least squares parameter estimator (6,7) for the model (3,A.i-iii). Assume*

$$\mathrm{Re}\left[(1 + e^{j\omega d} D(e^{j\omega}))^{-1} - 1/2\right] > 0 \; for \; 0 \leq \omega \leq 2\pi. \tag{66}$$

Suppose (51) holds. Then the adaptive predictor is self–optimizing in the sense of (52). If, $y(t)$, $u(t)$ and $w(t)$ are uniformly bounded, then the accumulated "regret" satisfies,

$$\sum_{t=1}^{N}(\hat{y}(t+d|t) - E(y(t+d)|\mathcal{F}_t))^2 = o(\log^2 N) \text{ a.s.} \tag{67}$$

Proof. As in Theorem 3.1, define $\alpha(t) := \dfrac{(\hat{y}(t+d|t) - E(y(t+d)|\mathcal{F}_t))^2}{1+\phi^T(t)P(t-d)\phi(t)}$. Then, from (11,25,49),

$$\begin{aligned}
&\textstyle\sum_{t=1}^{N}(\hat{y}(t+d|t) - E(y(t+d)|\mathcal{F}_t))^2 \\
&\le \textstyle\sum_{t=1}^{N} 2\alpha(t) + \sum_{t=1}^{N}\alpha(t)\phi^T(t)(P(t-d)-P(t))\phi(t) \\
&= 0(\log r_0(N)) + o(\log r_0(N)) \max_{1\le t\le N}\|\phi(t)\|^2 \text{ (since } P(t) \text{ converges)} \\
&= 0(\log r_0(N)) + o(\log r_0(N))[\max_{1\le t\le N}\|\phi(t)-\phi_0(t)\|^2 + \max_{1\le t\le N}\|\phi_0(t)\|^2] \\
&= 0(\log r_0(N)) + o(\log r_0^2(N)) + o(\log r_0(N))\max_{1\le t\le N}\|\phi_0(t)\|^2 \text{ (from (10))}.
\end{aligned}$$

The result (52) then follows from (51), while (67) follows from the boundedness of $\phi_0(t)$. □

Theorem 3.4. *Consider the direct adaptive predictor (49), using the extended stochastic gradient parameter estimator (6,7) for the model (3,A.i-iii). Assume*

$$\text{Re}\left[1 + e^{j\omega d}D(e^{j\omega})\right] > 0 \text{ for } 0 \le \omega \le 2\pi. \tag{68}$$

If (64) holds, then the adaptive predictor is self–optimizing in the sense of (52).

Proof. The result follows by using (49,40), and applying Kronecker's Lemma for the case $r(t) \to +\infty$, and then using (64). □

4 Indirect Adaptive Control

We need to reconsider the design of non–adaptive control laws. A general linear control law is

$$R(q^{-1})u(t) = S(q^{-1})y(t) + T(q^{-1})z(t) + M(q^{-1})w(t). \tag{69}$$

Here $z(t)$ is an exogenous signal, such as a command signal or a trajectory to be tracked. The control law above depends on $w(t)$, which is not available for measurement. However, we believe that this is the best starting point for the design of certainty–equivalent adaptive control laws, since an estimate $\widehat{w}(t)$ of $w(t)$ is automatically generated by the parameter estimation algorithm. (It is called $\widehat{v}(t)$ there). We will refer to indirect adaptive control based on (69) as "generalized certainty equivalence control."

We will impose the important restriction that,

$$R(q^{-1}) = R'(q^{-1})B(q^{-1}), \tag{70}$$

i.e., that $R(q^{-1})$ is divisible by $B(q^{-1})$.

When the control law (69,70) is applied to the system (1), the resulting closed–loop system is

$$y(t) = q^{-d}\frac{T(q^{-1})}{H(q^{-1})}z(t) + \frac{R'(q^{-1})C(q^{-1}) + q^{-d}M(q^{-1})}{H(q^{-1})}w(t), \tag{71}$$

where

$$H(q^{-1}) := A(q^{-1})R'(q^{-1}) - q^{-d}S(q^{-1}).$$

Above, we have cancelled the numerator polynomial $B(q^{-1})$. We will therefore require the following assumption.

$$\text{All the roots of } B(z) \text{ are outside the closed–unit disk.} \tag{72}$$

Additionally, since the design is intended to be stable, the closed–loop characteristic polynomial $H(q^{-1})$ should be chosen to be stable.

The closed–loop response (71) consists of two parts. The first part $q^{-d}\dfrac{T(q^{-1})}{H(q^{-1})}z(t)$ is the response to the command signal $z(t)$, while the second part $\dfrac{R'(q^{-1})C(q^{-1}) + q^{-d}M(q^{-1})}{H(q^{-1})}w(t)$ is the noise response.

Suppose there is a trajectory $y^*(t)$ to be tracked, given by

$$y^*(t) = \frac{q^{-d}B_m(q^{-1})}{A_m(q^{-1})}z(t),$$

i.e., the "reference model" is $\dfrac{q^{-d}B_m(q^{-1})}{A_m(q^{-1})}$. Then (see [24] for example), this is achieved by choosing $R'(q^{-1})$, $S(q^{-1})$ and $T(q^{-1})$ to satisfy,

$$A(q^{-1})R'(q^{-1}) - q^{-d}S(q^{-1}) = A_m(q^{-1})A_0(q^{-1}) \tag{73}$$

$$T(q^{-1}) = A_0(q^{-1})B_m(q^{-1}). \tag{74}$$

The polynomial $A_0(q^{-1})$ corresponds to an "observer," and can be chosen as any polynomial with roots outside the closed unit disk.

The Diophantine equation (73) has an infinite number of solutions for $R'(q^{-1})$ and $S(q^{-1})$. Let $(R'(q^{-1}), S(q^{-1}))$ be the unique solution for which

$$\deg(R'(q^{-1})) \leq d - 1. \tag{75}$$

Then $(R'(q^{-1}) + q^{-d}\Lambda(q^{-1}), S(q^{-1}) + A(q^{-1})\Lambda(q^{-1}))$ generates the set of all solutions as $\Lambda(q^{-1})$ is varied over the set of polynomials. Using this general solution yields the noise response $\dfrac{R'(q^{-1})C(q^{-1}) + q^{-d}(\Lambda(q^{-1})C(q^{-1}) + M(q^{-1}))}{H(q^{-1})}w(t)$. From this we see that there is freedom in choosing $\Lambda(q^{-1})$ and $M(q^{-1})$ to shape the noise response, even while maintaining the desired response $\dfrac{q^{-d}B_m(q^{-1})}{A_m(q^{-1})}z(t)$ to the command signal $z(t)$. In fact, we will simply take $\Lambda(q^{-1}) = 0$, since there is really no loss of freedom in exploiting just the choice of $M(q^{-1})$. Thus, the noise response to be shaped by the choice of $M(q^{-1})$ is,

$$\frac{R'(q^{-1})C(q^{-1}) + q^{-d}M(q^{-1})}{H(q^{-1})}w(t). \tag{76}$$

By choosing an appropriate noise shaping criterion, one can decide on $M(q^{-1})$. Let us consider two choices. The choice

$$M(q^{-1}) = -\text{ Truncation }(R'(q^{-1})C(q^{-1}), d) \tag{77}$$

minimizes the variance of $H(q^{-1})(y(t) - y^*(t))$. It results in the tracking error,

$$y(t) - y^*(t) = \frac{F_1(q^{-1})}{H(q^{-1})} w(t).$$

Above, $F_1(q^{-1}) := R'(q^{-1})C(q^{-1}) + q^{-d}M(q^{-1})$.

Alternatively, one may want to minimize the variance of the tracking error $(y(t) - y^*(t))$, while preserving the transfer function from the command signal $z(t)$ to the output $y(t)$ at its desired value $\dfrac{q^{-d}B_m(q^{-1})}{A_m(q^{-1})}$. Clearly, the variance of (76) is minimized when, $\dfrac{R'(q^{-1})C(q^{-1}) + q^{-d}M(q^{-1})}{H(q^{-1})} = F(q^{-1})$, for some polynomial $F(q^{-1})$ with deg $F(q^{-1}) \leq d-1$. A quick computation shows that $F(q^{-1})$ is actually the solution of (46). Also, with $G(q^{-1})$ as in (46), this choice of $M(q^{-1})$ is given by,

$$M(q^{-1}) = -R'(q^{-1})G(q^{-1}) - S(q^{-1})F(q^{-1}). \tag{78}$$

The resulting tracking error is

$$y(t) - y^*(t) = F(q^{-1})w(t).$$

There is one advantage in using the above model reference control law (69,70,73,75,74,46,78) over a minimum variance control which simply treats $y^*(t)$ as a target trajectory. Because the reference model dynamics is included as part of the closed loop poles (see (73)), the former may result in a smaller feedback gain and a smaller closed-loop bandwidth, and may therefore be more robust in the presence of the unmodeled dynamics.

The indirect adaptive control law based on (69) consists of first forming estimates $\widehat{A}(q^{-1}, t)$, $\widehat{B}(q^{-1}, t)$ and $\widehat{C}(q^{-1}, t)$ of the model (1). These are then used in place of $A(q^{-1})$, $B(q^{-1})$ and $C(q^{-1})$ in (73,75) and either (77) or (46,78) to obtain estimates $\widehat{R}'(q^{-1}, t)$,

$\widehat{S}(q^{-1}, t)$, $\widehat{M}(q^{-1}, t)$ of $R(q^{-1})$, $S(q^{-1})$ and $M(q^{-1})$. Finally, the estimate $\widehat{w}(t)$ of $w(t)$ provided by either of the algorithms (4,5) or (6,7) (denoted $\widehat{v}(t)$ there) is used in place of $w(t)$ in (69). We will call such a control law as a *generalized certainty equivalent* adaptive control law.

Theorem 4.1. *Consider the generalized certainty equivalent adaptive control law based on (69,70, 74,73,75,46,78,77) for the model (1,A.i-iii) with $d = 1$, using the extended least squares parameter estimator (4,5). Let us assume that,*

$$A_m(z) \text{ and } A_0(z) \text{ have all their roots outside the closed unit disk.} \tag{79}$$

Also assume the true system is strictly minimum phase, i.e., (72) holds,

$$b_0 \neq 0, \tag{80}$$

the positive real condition (50) holds, and

$$\{y^*(t)\} \text{ is uniformly bounded.} \tag{81}$$

Finally, suppose that

$$\liminf_{t\to\infty} |\widehat{b}_0(t)| > 0. \tag{82}$$

Then, for $\widehat{M}(q^{-1}, t)$ designed on the basis of (77), we have,

$$\sum_{t=1}^{N}[H(q^{-1})(y(t) - y^*(t)) - w(t)]^2 = O(N^\delta) \text{ a.s.} \quad \text{for every } \delta \in \left(\frac{2}{\alpha}, 1\right).$$

If $\widehat{M}(q^{-1}, t)$ is designed on the basis of (78), then

$$\sum_{t=1}^{N}[y(t) - y^*(t) - w(t)]^2 = O(N^\delta) \text{ a.s.} \quad \text{for every } \delta \in \left(\frac{2}{\alpha}, 1\right).$$

Proof. This proof is inspired by the work of [1]. Since $d = 1$, $F_1(q^{-1})$, $F(q^{-1})$, $R'(q^{-1})$, and their estimates are all just 1. Substituting the indirect adaptive version of the control law (69,70) into (54),

$$H(q^{-1})(y(t+1) - y^*(t+1)) = \widehat{M}(q^{-1}, t)\widehat{w}(t) + \widehat{C}(q^{-1}, t)\widehat{w}(t+1)$$
$$+C(q^{-1})(w(t+1) - \widehat{w}(t+1)) - \phi^T(t)\widetilde{\theta}(t).$$

When $\widehat{M}(q^{-1}, t)$ is designed via (77), this becomes

$$H(q^{-1})(y(t+1) - y^*(t+1)) = \hat{w}(t+1) + C(q^{-1})((w(t+1) - \hat{w}(t+1)) - \phi^T(t)\tilde{\theta}(t),$$

while if $\widehat{M}(q^{-1})$ is designed on the basis of (78), it becomes,

$$H(q^{-1})(y(t+1) - y^*(t+1)) \;\; = \;\; H(q^{-1})\hat{w}(t+1) + C(q^{-1})((w(t+1) - \hat{w}(t+1)) - \phi^T(t)\tilde{\theta}(t).$$

With these representations of the dynamics as the starting point, the rest of the proof is completed as in [1], with only slight modifications to account for the presence of the term $H(q^{-1})$ in the LHS above. □

Theorem 4.2. *Consider the generalized certainty equivalent adaptive control law based on (69,70, 74,73,75,46,78,77) for the model (1,A.i-iii), using the extended stochastic gradient parameter estimator (6, 7). Assume (72,80,63,79,81) and*

$$\text{the distribution of } w(t) \text{ is mutually absolutely continuous with respect to Lebesgue measure for all } t \geq 0. \tag{83}$$

Then, for the design (77), we have,

$$\lim_{N\to\infty} 1/N \sum_{t=1}^{N} [H(q^{-1})(y(t) - y^*(t)) - F_1(q^{-1})w(t)]^2 = 0 \qquad \text{a.s.}$$

while for the design (78),

$$\lim_{N\to\infty} 1/N \sum_{t=1}^{N} [y(t) - y^*(t) - F(q^{-1})w(t)]^2 = 0 \qquad \text{a.s.}$$

Proof. Let us consider the non-trivial case $r(t) \to +\infty$. By the minimum phase assumption, $r_0(N-1) = O(N) + O\left(\sum_{t=1}^{N} y^2(t)\right)$. Substituting the indirect adaptive version of the control law (69,70) into (55), and then proceeding as in Theorem 3.2, we obtain

$$\lim_{N\to\infty} \tfrac{1}{r_0(N-1)} \textstyle\sum_{t=1}^{N} [H(q^{-1})(y(t) - y^*(t)) - (\hat{R}'(q^{-1}, t)\hat{C}(q^{-1}, t) + q^{-d}\widehat{M}(q^{-1}, t))w(t)]^2 = 0 \;\; \text{a.s.} \tag{84}$$

Hence,

$$\sum_{t=1}^{N}[H(q^{-1})(y(t)-y^*(t))-(\widehat{R}'(q^{-1},t)\widehat{C}(q^{-1},t)+q^{-d}\widehat{M}(q^{-1},t))w(t)]^2 = o(N)+o\left(\sum_{t=1}^{N} y^2(t)\right) \quad \text{a.s.}$$

Since $\widehat{R}'(q^{-1},t)$, $\widehat{C}(q^{-1},t)$, $\widehat{M}(q^{-1},t)$ and $y^*(t)$ are bounded uniformly, we obtain $\sum_{t=1}^{N} y^2(t) = O(N)$. This yields, $r_0(N) = O(N)$ a.s. Hence from (84),

$$\lim_{N\to\infty} 1/N \sum_{t=1}^{N}[H(q^{-1})(y(t)-y^*(t))-(\widehat{R}'(q^{-1},t)\widehat{C}(q^{-1},t)+q^{-d}\widehat{M}(q^{-1},t))w(t)]^2 = 0 \quad \text{a.s.}$$

If $\widehat{M}(q^{-1},t)$ is based on (77), then

$$\lim_{N\to\infty} 1/N \sum_{t=1}^{N}[H(q^{-1})(y(t)-y^*(t))-\widehat{F}_1(q^{-1},t)w(t)]^2 = 0 \quad \text{a.s.},$$

while if $\widehat{M}(q^{-1},t)$ is based on (78),

$$\lim_{N\to\infty} 1/N \sum_{t=1}^{N}[H(q^{-1})(y(t)-y^*(t)-\widehat{F}(q^{-1},t)w(t))]^2 = 0 \quad \text{a.s.}$$

Noting that $H(q^{-1})y(t) - F_1(q^{-1})w(t)$ and $H(q^{-1})(y(t) - F(q^{-1})w(t))$ are both $\mathcal{F}_{t-d}$-measurable, the rest of the argument to complete the proof is the same as in (65). □

5 Direct Adaptive Control

The starting point here is the parametrization of the system as in (3). If $y^*(t)$ is the target trajectory to be tracked, it is clear that the minimum variance tracking control law is

$$R(q^{-1})u(t) = y^*(t+d) - S(q^{-1})y(t) - D(q^{-1})v(t).$$

The certainty–equivalent version of this simply replaces $R(q^{-1})$, $S(q^{-1})$, $D(q^{-1})$ and $v(t)$ by their estimates as given by either the algorithms (4,5) or (6,7). The resulting adaptive control law is

$$\phi^T(t)\widehat{\theta}(t) = y^*(t+d). \tag{85}$$

Direct adaptive model reference control which attempts to achieve

$$y(t+d) - \frac{B_m(q^{-1})}{A_m(q^{-1})} y^*(t+d) = v(t+d) \tag{86}$$

can be constructed based on the following non-adaptive design.

Lemma 5.1. *Assume (79,72). Then the model reference control which asymptotically achieves (86) is given by*

$$\begin{aligned} W(q^{-1})R(q^{-1})u(t) + (W(q^{-1})S(q^{-1}) + T(q^{-1}))y(t) + (W(q^{-1})D((q^{-1})) - T(q^{-1}))v(t) \\ = B_m(q^{-1})A_0(q^{-1})y(t+d), \end{aligned} \tag{87}$$

where

$$\begin{aligned} W(q^{-1}) &= Truncation(A_m(q^{-1})A_0(q^{-1}), d) \\ T(q^{-1}) &= q^d(A_m(q^{-1})A_0(q^{-1}) - W(q^{-1}) \end{aligned} \tag{88}$$

The resulting closed loop characteristic polynomial is $A_m(q^{-1})A_0(q^{-1})B(q^{-1})$. (89)

Proof. Multiplying (3) by W and using (88), we get

$$A_m A_0 y(t+d) \;\; = \;\; WRu(t) + (WS+T)y(t) + WDv(t).$$

Substituting the control law (87) into the above yields (86).

The closed loop characteristic polynomial is obtained by looking at the feedback connection of the plant transfer function $q^{-d}\frac{B}{A}$ and the controller transfer function $-\frac{WS+T}{WR}$, which is $AWR + q^{-d}B(WS+T)$. Simplification then yields (89). □

Remark: The fact (89) indicates that (87) is indeed a "genuine" model reference controller, instead of a simple minimum variance controller in disguise.

From (87), a direct adaptive model reference control is then

$$W(q^{-1})(\phi^T(t)\widehat{\theta}(t)) = B_m(q^{-1})A_0(q^{-1})y^*(t+d) - T(q^{-1})(y(t) - \widehat{v}(t)) \tag{90}$$

Theorem 5.1. *Consider the direct adaptive control law (90) for the model (3,A.i-iii)), using the extended least squares parameter estimator (4,5). Assume (72,80,66), and*

$$y^*(t) \text{ is a uniformly bounded deterministic sequence.} \tag{91}$$

Suppose that

$$\liminf_{t\to\infty} |\hat{r}_0(t)| > 0, \tag{92}$$

then

$$\sum_{t=1}^{N}(y(t) - \frac{B_m(q^{-1})}{A_m(q^{-1})}y^*(t) - F(q^{-1})w(t))^2 = O(N^\delta) \quad \text{a.s.} \quad \text{for every } \delta \in \left(\frac{2}{\alpha}, 1\right).$$

Proof. Rewrite (90) as

$$\begin{aligned} &W(q^{-1})R(q^{-1})u(t) + W(q^{-1})S(q^{-1})y(t) + W(q^{-1})D((q^{-1}))\hat{v}(t) \\ &= -W(q^{-1})(\phi^T(t)\tilde{\theta}(t)) + B_m(q^{-1})A_0(q^{-1})y^*(t+d) - T(q^{-1})(y(t) - \hat{v}(t)). \end{aligned}$$

Combining (3) and the above yields

$$A_m A_0 y(t+d) - B_m A_0 y^*(t+d) - A_m A_0 v(t+d) = T[\hat{v}(t) - v(t)] - WD(\hat{v}(t) - v(t)] - W(\phi^T(t)\tilde{\theta}(t)). \tag{93}$$

The rest of the proof proceeds similarly to [1] with the definition $\alpha(t) := \frac{(y(t)-y^*(t)-v(t))^2}{1+\phi^T(t-d)P(t-2d)\phi(t-d)}$.

□

Theorem 5.2. *Consider the direct adaptive control law (90) for the model (3,A.i-iii)), using the extended stochastic gradient parameter estimator (6,7). Assume (72,80,68,83,91). Then*

$$\lim_{N\to\infty} 1/N \sup_N \sum_{t=1}^{N}(y(t) - y^*(t) - F(q^{-1})(t))^2 = 0 \quad \text{a.s.}$$

Proof. The results follow from (41,93), using the Key Technical Lemma of [4]. □

6 Convergence of Adaptive Controllers

In the previous sections we have established the self–optimality of some indirect and direct adaptive control algorithms. We now address the issue of convergence of the parameter estimates and the adaptive controller coefficients. We introduce a general method for proceeding from self–optimality to the establishment of such self–tuning properties.

To exhibit the method, we consider two adaptive control laws. The first is the indirect adaptive control using the SG algorithm, and the second is the direct adaptive control using the ELS algorithm.

Theorem 6.1. *Consider the indirect adaptive control law (69,70, 74,73,75,46,78) for the model (1,A.i-iii), using the extended stochastic gradient parameter estimator (6,7). Assume the conditions of Theorem 4.2. Define* $y^0(t)$ *as,*

$$y^0(t) := z(t) + F(q^{-1})w(t), z(t) := \frac{B_m(q^{-1})}{A_m(q^{-1})}y^*(t), \tag{94}$$

and $u^0(t)$ *by,*

$$A(q^{-1})y^0(t) = q^{-d}B(q^{-1})u^0(t) + C(q^{-1})w(t). \tag{95}$$

Let $\phi^0(t)$ *and* $R^0(t)$ *be defined as* $\phi_0(t)$ *and* $R_0(t)$*, except that* $y(t)$ *and* $u(t)$ *are replaced by* $y^0(t)$ *and* $u^0(t)$*. Suppose that the ensemble autocorrelations of* $\{y^*(t)\}$ *exist. Define*

$$\Phi^0 := \lim_{N\to\infty} R^0(N-1)/N \qquad \text{a.s.}$$

Then

(i) $\widetilde{\theta}(t) \to \mathcal{N}(\Phi^0)$ a.s., *where* $\mathcal{N}(\Phi^0) :=$ *null space of* Φ^0.

(ii) For all $\widetilde{\theta} \in \mathcal{N}(\Phi^0)$*, let* $\widehat{A}(q^{-1})$, $\widehat{B}(q^{-1})$, $\widehat{C}(q^{-1})$, $\widehat{F}(q^{-1})$ *and* $\widehat{G}(q^{-1})$ *be the polynomials corresponding to* $\widetilde{\theta} + \theta^0$*. Then*

$$\widehat{F}(q^{-1}) = F(q^{-1}) \tag{96}$$

and

$$\widehat{B}(q^{-1})G(q^{-1}) = B(q^{-1})\widehat{G}(q^{-1}). \tag{97}$$

(iii) If $B(q^{-1})$ and $G(q^{-1})$ do not have a common factor, and $\{z(t)\}$ is persistently exciting of order ℓ_p, with

$$\ell_p \geq \min(\deg C(q^{-1}) - \deg F(q^{-1}) + 1, \deg A(q^{-1}) + 1),$$

then

$$\lim_{t\to\infty} \widehat{\theta}(t) = \theta^0 \qquad \text{a.s.}$$

Proof. From Theorem 4.2, $\lim_{N\to\infty} 1/N \sum_{t=0}^{N}(y(t) - y^o(t))^2 = 0$ a.s.. Since $B(q^{-1})$ is strictly minimum phase, and $B(q^{-1})(u^0(t) - u(t)) = A(q^{-1})(y^0(t+d) - y(t+d))$, it follows that $\lim_{N\to\infty} 1/N \sum_{t=0}^{N}(u(t) - u^o(t))^2 = 0$ a.s.. Hence,

$$1/N \sum_{t=0}^{N} \|\phi^o(t) - \phi(t)\|^2 \to 0 \text{ a.s.}$$

and

$$1/N\|R^o(N) - R(N)\| \to 0 \text{ a.s.}$$

It then follows from (42) that

$$1/N\|R^o(N-1)\widetilde{\theta}(N)\| \to 0 \text{ a.s.}$$

Therefore, $\lim_{N\to\infty} \|\Phi^0\widetilde{\theta}(N)\| = 0$ a.s. This proves that $\widetilde{\theta}(N) \to \mathcal{N}(\Phi^0)$ a.s.

Consider now a vector $\widetilde{\theta} \in \mathcal{N}(\Phi^0)$. We have

$$\lim_{N\to\infty} 1/N \sum_{t=1}^{N}[(A(q^{-1}) - \widehat{A}(q^{-1}))y^0(t) + q^{-d}(\widehat{B}(q^{-1}) - B(q^{-1}))u^0(t) + (\widehat{C}(q^{-1}) - C(q^{-1}))w(t)]^2 = 0 \quad \text{a.s.}$$

Since $y^0(t) = z(t) + F(q^{-1})w(t)$, we obtain

$$\begin{aligned} &\lim_{N\to\infty} 1/N \sum_{t=1}^{N} \Big[(A(q^{-1}) - \widehat{A}(q^{-1}))z(t) + q^{-d}(\widehat{B}(q^{-1}) - B(q^{-1}))u^0(t) \\ &\quad + ((A(q^{-1}) - \widehat{A}(q^{-1}))F(q^{-1}) + \widehat{C}(q^{-1}) - C(q^{-1}))w(t)\Big]^2 = 0 \text{ a.s.} \end{aligned} \tag{98}$$

Now note that the term involving $u^0(t)$ above is $\mathcal{F}_{t-d}$-measurable. Hence, proceeding as in [25], it follows that the first d coefficients of $(A(q^{-1}) - \widehat{A}(q^{-1}))F(q^{-1}) + \widehat{C}(q^{-1}) - C(q^{-1})$ are zero, i.e., $\widehat{A}(q^{-1})F(q^{-1}) - \widehat{C}(q^{-1})$ is divisible by q^{-d}. This implies (96).

Multiplying the summand in (98) by $B(q^{-1})$ and using (95) gives

$$\lim_{N\to\infty} 1/N \Big[(\widehat{B}(q^{-1})A(q^{-1}) - B(q^{-1})\widehat{A}(q^{-1}))z(t) + B(q^{-1})(\widehat{C}(q^{-1}) - \widehat{A}(q^{-1})F(q^{-1})) \\ -\widehat{B}(q^{-1})(C(q^{-1}) - A(q^{-1})F(q^{-1}))\, w(t)\big]^2 = 0 \text{ a.s.}$$

Since $z(t)$ is deterministic, we can proceed as in [25], to get

$$\lim 1/N \sum_{t=1}^{N} \Big[(\widehat{B}(q^{-1})A(q^{-1}) - B(q^{-1})\widehat{A}(q^{-1}))z(t) \Big]^2 = 0 \text{ a.s.} \tag{99}$$

as well as

$$\lim_{N\to\infty} 1/N \textstyle\sum_{t=1}^{N} [(B(q^{-1})(\widehat{C}(q^{-1}) - \widehat{A}(q^{-1})F(q^{-1})) - \widehat{B}(q^{-1})(C(q^{-1}) - A(q^{-1})F(q^{-1})))w(t)]^2 \\ = 0 \text{ a.s.}$$

This yields

$$B(q^{-1})(\widehat{C}(q^{-1}) - \widehat{A}(q^{-1})F(q^{-1})) = \widehat{B}(q^{-1})(C(q^{-1}) - A(q^{-1})F(q^{-1})), \tag{100}$$

which in turn yields (97).

Finally, to prove (iii), note that if $B(q^{-1})$ and $G(q^{-1})$ do not have a common factor, then $\widehat{B}(q^{-1}) = \lambda B(q^{-1})$ and $\widehat{G}(q^{-1}) = \lambda G(q^{-1})$ for some common scalar λ. Hence, from (99), $\lim\limits_{N\to\infty} 1/N \sum\limits_{t=1}^{N} [B(q^{-1})(\lambda A(q^{-1}) - \widehat{A}(q^{-1}))z(t)]^2 = 0$ a.s. By (72), this reduces to,

$$\lim_{N\to\infty} 1/N \sum_{t=1}^{N} [(\lambda A(q^{-1}) - \widehat{A}(q^{-1}))z(t)]^2 = 0 \text{ a.s.} \tag{101}$$

If $\ell_p \geq 1 + \deg A(q^{-1})$, then from the above, $\lambda A(q^{-1}) = \widehat{A}(q^{-1})$, which yields $\lambda = 1$, and thus also $\widehat{C}(q^{-1}) = C(q^{-1})$.

From (100), we obtain

$$(\widehat{B}A - B\widehat{A})F = \widehat{B}C - B\widehat{C}\,.$$

Therefore,

$$\lambda A - \widehat{A} = \frac{\lambda C - \widehat{C}}{F}$$

It follows that unless $\lambda = 1$, $\deg(\lambda A - \widehat{A}) = \deg(\lambda C - \widehat{C}) - \deg F \leq \deg C - \deg F$. Hence, if $\ell_p \geq \deg C - \deg F + 1$, we can again use (101) to conclude parameter consistency. $\square$

Theorem 6.2. *Consider the direct adaptive control law (90) for the model (3,A.i-iii), using the extended least squares parameter estimator (4,5). Assume the conditions of Theorem 5.1. Let $y^0(t)$, $z(t)$ and $u^0(t)$ be as in Theorem 6.1. Let $\phi^0(t)$ and $R^0(t)$ be defined as $\phi_0(t)$ and $R_0(t)$, except that $y(t)$ and $u(t)$ are replaced by $y^0(t)$ and $u^0(t)$. Suppose that the ensemble autocorrelations of $\{z(t)\}$ exist and satisfy,*

$$1/N \sum_{t=1}^{N} \left[z(t)^2 - \lim_{M \to \infty} 1/M \sum_{k=1}^{M} z(t)^2 \right] = 0(N^{-\delta}) \quad \text{for some } \delta > 0.$$

Define

$$\Phi^0 := \lim_{N \to \infty} R^0(N-1)/N \qquad \text{a.s.}$$

Then

(i) $\widetilde{\theta}(t) \to \mathcal{N}(\Phi^0)$ a.s., *where* $\mathcal{N}(\Phi^0) :=$ *null space of* Φ^0.

(ii) For all $\widetilde{\theta} \in \mathcal{N}(\Phi^0)$, let $\widehat{R}(q^{-1})$, $\widehat{S}(q^{-1})$, and $\widehat{D}(q^{-1})$ be the polynomials corresponding to $\widetilde{\theta} + \theta^0$. Then

$$\widehat{R}(q^{-1})(S(q^{-1}) + D(q^{-1})) = R(q^{-1})(\widehat{S}(q^{-1}) + \widehat{D}(q^{-1})). \tag{102}$$

(iii) If $R(q^{-1})$ and $S(q^{-1}) + D(q^{-1})$ do not have a common factor, and $\{z(t)\}$ is persistently exciting of order ℓ_p, with

$$l_p \geq 2 + \min(\deg S(q^{-1}), \deg D(q^{-1})),$$

then

$$\lim_{t \to \infty} \widehat{\theta}(t) = \theta^0 \qquad \text{a.s.}$$

Proof. From Theorem 5.1, $\sum_{t=1}^{N}(y(t) - y^0(t))^2 = O(N^\delta)$ a.s. for some $1/2 < \delta < 1$. Since $B(q^{-1})$ is strictly minimum phase, and $B(q^{-1})(u^0(t) - u(t)) = A(q^{-1})(y^0(t+d) - y(t+d))$, it follows that $\sum_{t=1}^{N}(u(t) - u^0(t))^2 = O(N^\delta)$ a.s. Hence $\sum_{t=1}^{N}\|\phi^0(t) - \phi(t)\|^2 = O(N^\delta)$ a.s. for some $1/2 < \delta < 1$. Now note that from Theorem 2.1, $\tilde{\theta}^T(N)R(N-1)\tilde{\theta}(N) = O(\log\lambda_{\max}(R(N-1))$. Hence $\tilde{\theta}^T(N)\dfrac{R(N-1)}{N}\tilde{\theta}(N) = O\left(\log N/N\right) = o(1)$. Note also that

$$\begin{aligned}
\left\|R^0(N-1) - R(N-1)/N\right\| &= 1/N\left\|\sum_{t=1}^{N}\phi^0(t)(\phi^0(t)-\phi(t))^T + (\phi^0(t)-\phi(t))\phi^T(t)\right\| \\
&= O\left(1/N\left(\textstyle\sum_{t=1}^{N}\|\phi^0(t)\|^2 + \|\phi(t)\|^2\right)^{1/2}\left(\textstyle\sum_{t=1}^{N}\|\phi^0(t)-\phi(t)\|^2\right)^{1/2}\right) \\
&= O\left(N^{\delta-1/2}\right) \text{ for some } 0<\delta<1/2.
\end{aligned}$$

Hence

$$\begin{aligned}
\tilde{\theta}^T(N)\frac{R^0(N-1)}{N}\tilde{\theta}(N) &= \tilde{\theta}^T(N)\frac{R(N-1)}{N}\tilde{\theta}(N) + \tilde{\theta}^T(N)\frac{(R^0(N-1)-R(N-1)}{N}\tilde{\theta}(N) \\
&= O\left(\log N/N\right) + O\left(\frac{\log N}{N^{1/2-\delta}}\right).
\end{aligned}$$

Noting that $\left\|R^0(N-1)/N - \Phi^0\right\| = O(N^{-\delta})$ for some $\delta > 0$ and that $\|\tilde{\theta}(N)\|^2 = O(logN)$, we obtain $\lim_{N\to\infty}\tilde{\theta}^T(N)\Phi^0\tilde{\theta}(N) = 0$ a.s. This proves that $\tilde{\theta}(N) \to \mathcal{N}(\Phi^0)$ a.s.

Consider now a vector $\tilde{\theta} \in \mathcal{N}(\Phi^0)$. We have

$$\lim_{N\to\infty} 1/N\sum_{t=1}^{N}[\tilde{R}(q^{-1})y^0(t) + \tilde{S}(q^{-1})u^0(t) + \widetilde{D}(q^{-1}))v(t)]^2 = 0 \quad \text{a.s.}$$

Substituting (94,95) into the above yields

$$\lim_{N\to\infty} 1/N\sum_{t=1}^{N}[(q^d\tilde{R} + \hat{S}R - \hat{R}S)z(t) + ((\hat{S}+\widehat{D})R - (S+D)\hat{R})v(t)]^2 = 0 \quad \text{a.s.}$$

The rest of the proof resembles that of Theorem 6.1. □

7 Concluding Remarks

We have provided here a start towards a more comprehensive asymptotic theory of stochastic adaptive systems, showing an analysis of several adaptive filtering, control and identification algorithms, employing either an indirect or a direct procedure, and either a least–squares or a gradient based parameter estimator. We have shown how to analyze non–interlaced algorithms, which are particularly useful for direct adaptive prediction and control. We have also proposed a generalized certainty equivalence approach which provides a basis for separating the servo and regulation problems. This allows the design of algorithms for model reference control with simultaneous disturbance rejection. We have also shown that in general the parameter estimates converge to the null space of a certain matrix. This allows us to establish the convergence of several adaptive controllers.

Several outstanding problems remain. For indirect adaptive model reference control using the ELS algorithm, the analysis of multiple delay systems is still open, and the assumption that the estimate $\widehat{b}_0(t)$ stays away from zero can not be easily verified or dispensed with. There are also several open problems when one moves away from model reference schemes. In particular, the treatment of non–minimum phase systems, and the attendant issue of possible uncontrollability of the estimated model, still eludes a satisfactory solution.

Finally, there is another potential of a stochastic formulation which is very much unrealized. This is the issue of how to improve transient performance by properly incorporating any prior knowledge of the system. A way to do this is to employ a prior distribution for the system parameters, and then to resort to a Bayesian solution. However, it should be noted that this approach requires the modeling of the entire probability distribution of the disturbance, and not just its mean value, as done here. The optimal solution of this dual control problem is now regarded as intractable. But several questions remain even for a sub–optimal solution. How does it perform for a particular system? How robust is the resulting controller to mis–specification of the noise distributions? Such a theory needs future attention.

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