Improved Capacity Bounds for Wireless Networks*

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Abstract

We obtain improved upper and lower bounds on the best case and random case transport capacities of wireless networks under the Protocol Model of communication in [1]. These results bracket the best case transport capacity to within a factor of $\sqrt{8}$ for wireless networks on a disk. This is done by identifying larger exclusion regions for receivers. The general result on exclusion regions can also be applied to arbitrary wireless footprints, including those arising from directional antennas, thus obtaining superior bounds for such technologies too.

1 Introduction

Bounds on the capacity of wireless networks were investigated in [1]. It considers a wireless network formed by n nodes in a disc of unit area. Two

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models of communication were considered. The *Protocol Model* requires that each receiver lie outside the interference region of every other transmitter. Here each transmitter transmitting to a receiver at distance r forms an interference region consisting of a disc of radius $(1+\Delta)r$ centered around itself. Under the alternate *Physical Model* of communication, the signal to interference plus noise ratio (SINR) is required to be above a pre-specified threshold β, where the signal power is assumed to decay as $r^{-\alpha}$ with distance r, for $\alpha > 2$.

The paper [1] considers the *transport capacity*, defined as the total bit-distance product per sec that can be transported in the network. In the best case, under the Protocol Model of communication, upper and lower bounds of $\sqrt{\frac{8}{\pi}}\frac{1}{\Delta}W\sqrt{n}$ bit-meters/sec and $\frac{W}{1+2\Delta}\frac{n}{\sqrt{n}+\sqrt{8\pi}}$ bit-meters/sec, respectively, were obtained, where W is the capacity of the wireless channel in bits/sec. For random networks, i.e., networks where the node locations are uniformly and independently distributed, and a destination is randomly chosen for each node, the upper bound on the per node throughput capacity was shown to be $\frac{c''W}{\Delta^2\sqrt{n\log n}}$ under the Protocol Model.

The idea used in [1] was to show that discs of size $\frac{\Delta r}{2}$ around the receivers are *exclusion regions*, i.e., are mutually disjoint from each other, where r is the distance from the transmitter to the particular receiver. In this paper, we improve the bounds by showing larger exclusion regions. In particular, for the Protocol Model, we show that the exclusion region for each transmission is an ellipse along the transmitter-receiver axis with the transmitter and the receiver being at the two foci.

We further explore the issue of determining exclusion regions for arbitrary interference patterns and receiver configurations. We identify exclusion regions in a general way and show that for a single receiver case, such regions are always convex. We specifically illustrate the exclusion regions for different interesting cases.

The main utilitarian aspect of our contribution is that, using the concept of exclusion regions, we tighten the capacity bounds obtained in [1]. In particular, we improve the upper bounds for the best and random network cases to $\sqrt{\frac{8}{\pi}}\frac{W\sqrt{n}}{\sqrt{(1+\Delta)\sqrt{\Delta}\sqrt{2+\Delta}}}$ bit-meters/sec and $\frac{c''W}{(1+\Delta)\sqrt{\Delta}\sqrt{2+\Delta}\sqrt{n\log n}}$ bit-meters/sec respectively. We also show that by using a *grid of ellipses* configuration, one can improve the lower bound for the best case to $\sqrt{\frac{1}{\pi}}\frac{W\sqrt{n}}{\sqrt{(1+\Delta)\sqrt{\Delta}\sqrt{2+\Delta}}}$ bit-meters/sec. The interesting point to note is that, by taking the ratio

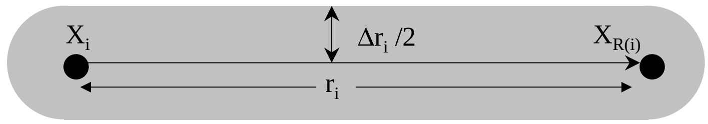

Figure 1: An Exclusion Region for the Protocol Model.

of the upper bound on the best case to the above identified feasible transport capacity, the best achievable transport capacity is bracketed to within a factor of $\sqrt{8} = 2.83$, irrespective of Δ, thus characterizing it fairly sharply. Our results can also be applied to directional antennas, and we illustrate the exclusion regions for such technologies.

2 Network Model and Exclusion Regions

Consider a network of n nodes in a disc of unit area. Let X_i, $1 \le i \le n$, denote the location of node i. We will use X_k to denote a node as well as its location. Let $\{(X_k, X_{R(k)}) : k \in T\}$ be the set of all active transmitter-receiver pairs in some particular slot. As in [1], we assume a slotted model for convenience of exposition. Let the transmission radius, $|X_k - X_{R(k)}|$, be denoted as r_k. We first describe the Protocol Model of communication.

2.1 The Protocol Model

The transmission from node X_i, $i \in T$, is successfully received by the receiver $X_{R(i)}$ only if

$$|X_k - X_{R(i)}| \ge (1+\Delta)|X_i - X_{R(i)}|, \quad (1)$$

for every $k \in T \backslash i$. Here $\Delta > 0$ models a guard zone around the transmission region.

2.2 Exclusion Regions

In [1], it is shown that the $\frac{\Delta r}{2}$ neighborhood around a receiver of a transmission of range r is an *exclusion region.* First, we define the notion of an exclusion region.

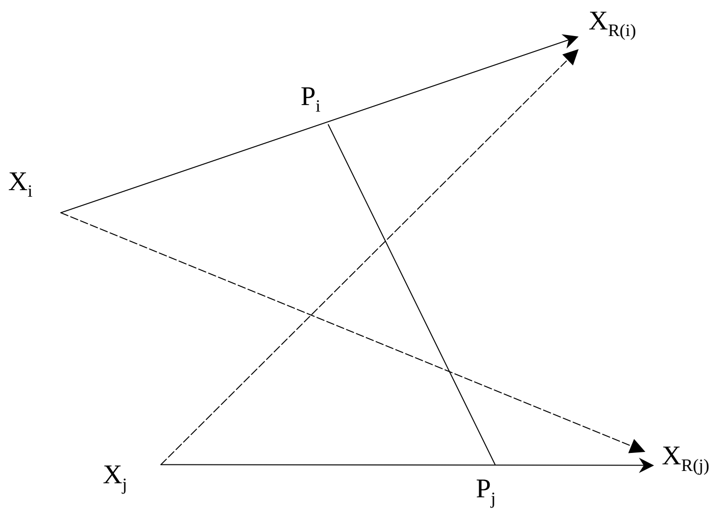

Figure 2: A Transmitter-Receiver Pair in the Protocol Model.

Definition 2.1 *An exclusion region is an area associated with an active receiver that must remain disjoint from the exclusion region of every other active receiver in the network at that time, for any configuration of transmitters and receivers.*

We will now show that there is a capsule-shaped region around the transmitter-receiver axis that is an exclusion region for the Protocol Model, as shown in Figure 1.

Theorem 2.1 *In a wireless network, under the Protocol Model of communication, the $\frac{\Delta r_i}{2}$ neighborhood of the line joining X_i and $X_{R(i)}$ is an exclusion region, where $r_i := |X_i - X_{R(i)}|$.*

Proof: Let $(X_i, X_{R(i)})$ and $(X_j, X_{R(j)})$ be any two separate transmitter-receiver pairs. Let P_i and P_j be any points on the transmitter-receiver axes respectively, as shown in Figure 2. Applying the triangle inequality,

$$|X_i - P_i| + |P_i - P_j| + |P_j - X_{R(j)}| \geq |X_i - X_{R(j)}|, \tag{2}$$

and

$$|X_j - P_j| + |P_j - P_i| + |P_i - X_{R(i)}| \geq |X_j - X_{R(i)}|. \tag{3}$$

Adding (2) and (3) we get

$$\begin{aligned} & |X_i - P_i| + |P_i - X_{R(i)}| + |X_j - P_j| + |P_j - X_{R(j)}| + 2|P_j - P_i| \\ \geq \ & |X_i - X_{R(j)}| + |X_j - X_{R(i)}|, \end{aligned}$$

which reduces since, $|X_i - P_i| + |P_i - X_{R(i)}| = |X_i - X_{R(i)}|$, to

$$|X_i - X_{R(i)}| + |X_j - X_{R(j)}| + 2|P_j - P_i| \geq |X_i - X_{R(j)}| + |X_j - X_{R(i)}|.$$

Further, applying the constraint (1), we get

$$|X_i - X_{R(i)}| + |X_j - X_{R(j)}| + 2|P_j - P_i| \geq (1+\Delta)(|X_i - X_{R(i)}| + |X_j - X_{R(j)}|),$$

which simplifies to

$$|P_j - P_i| \geq \frac{\Delta(r_i + r_j)}{2}. \tag{4}$$

It is now easy to see that the $\frac{\Delta r_i}{2}$ and $\frac{\Delta r_j}{2}$ open neighborhoods of the segments $(X_i, X_{R(i)})$ and $(X_j, X_{R(j)})$, respectively, are disjoint. To see why, suppose there is a point P that is contained in both the neighborhoods. Let P_i and P_j respectively be the points on the segments $(X_i, X_{R(i)})$ and $(X_j, X_{R(j)})$ closest to such a P. We then have

$$\begin{aligned} |P - P_i| &< \frac{\Delta r_i}{2}, \text{ and} \\ |P - P_j| &< \frac{\Delta r_j}{2}. \end{aligned}$$

Applying the triangle inequality, we get

$$\begin{aligned} |P_i - P_j| &< |P - P_i| + |P - P_j| \\ &< \frac{\Delta r_i}{2} + \frac{\Delta r_j}{2} \\ &= \frac{\Delta(r_i + r_j)}{2}, \end{aligned}$$

which contradicts (4). □

Figure 1 shows the exclusion region for the case of two dimensions. The area of this region, $\Delta r_i^2(1 + \frac{\pi\Delta}{4})$, is much larger than that of the disc of radius $\frac{\Delta r}{2}$ identified in [1], which is $\frac{\pi\Delta^2 r_i^2}{4}$. Using the area of this region improves the upper bounds on capacity in [1]. However, we will show next how to obtain an even larger exclusion region using a very general approach which also applies to other contexts of interest, such as directional antennas.

2.3 Generalized Protocol Model for General Interference Regions

As before, let $\{(X_k, X_{R(k)}) : k \in T\}$ denote the set of all active transmitter receiver pairs. Associated with each transmitter-receiver pair, $(X_k, X_{R(k)})$, $k \in T$, let there be an *interference region*, I_k, and assume that $X_{R(k)} \in I_k$. Suppose also that the necessary condition for a transmission from X_k to $X_{R(k)}$ to be successful is

$$X_{R(k)} \notin I_j, \forall j \in T, j \neq k. \tag{5}$$

Also no node can be simultaneously a receiver and a transmitter. Such a general interference footprint I_k can be used to model, for example, directional antennas, as we will do in Section 4.

2.4 Exclusion Region for the Generalized Protocol Model

We will now show how to construct exclusion regions for such a general model. Note that the earlier Protocol Model of communication corresponds to the special case where I_k is a disk of radius $(1+\Delta)|X_i - X_{R(i)}|$ centered at X_i.

It is easy to see that a simple exclusion region is a disk centered at $X_{R(i)}$ with radius $\frac{\gamma}{2}$, where $\gamma = \inf_{P \notin I_k} |X_{R(k)} - P|$. However, we propose a larger exclusion region:

Theorem 2.2 *The set*

$$E_k = \{P : |P - X_{R(k)}| < |P - Q|, \forall Q \notin I_k\} \tag{6}$$

is a valid exclusion region for each receiver $R(k)$*.*

Proof: If not, let $E_i \cap E_j \neq \phi$ for some distinct $i, j \in T$, and let $P \in E_i \cap E_j$. Then we have

$$\begin{aligned} |P - X_{R(i)}| &< |P - Q|, \forall Q \notin I_i, \text{ and} & (7)\\ |P - X_{R(j)}| &< |P - Q|, \forall Q \notin I_j. & (8) \end{aligned}$$

Also, using (5), we have

$$\begin{aligned} X_{R(i)} &\notin I_j, \text{ and} & (9)\\ X_{R(j)} &\notin I_i. & (10) \end{aligned}$$

Using (7) and (10), and setting $Q = X_{R(j)}$, we get

$$|P - X_{R(i)}| < |P - X_{R(j)}|. \tag{11}$$

Similarly, setting $Q = X_{R(i)}$ in (8), due to (9) we get,

$$|P - X_{R(j)}| < |P - X_{R(i)}|. \tag{12}$$

The strict inequalities (11) and (12) lead to a contradiction which completes the proof. □

Next we establish an important property of the exclusion regions defined above.

Theorem 2.3 *The exclusion region E_k, defined in (6), is a convex set.*

Proof: If not, let $P_1, P_3 \in E_k$, and $P_2 \notin E_k$ with P_2 lying on the line segment (P_1, P_3). Using (6), we have

$$\begin{aligned} |P_1 - X_{R(k)}| &< |P_1 - Q|, \forall Q \notin I_k, \text{ and} \quad (13)\\ |P_3 - X_{R(k)}| &< |P_3 - Q|, \forall Q \notin I_k, \quad (14) \end{aligned}$$

while for some $Q_2 \notin I_k$,

$$|P_2 - X_{R(k)}| \geq |P_2 - Q_2|. \tag{15}$$

Setting $Q = Q_2$ in (13) and (14), we get

$$\begin{aligned} |P_1 - X_{R(k)}| &< |P_1 - Q_2|, \text{ and} \quad (16)\\ |P_3 - X_{R(k)}| &< |P_3 - Q_2|. \quad (17) \end{aligned}$$

Let $\Gamma = \{P : |P - X_{R(k)}| < |P - Q_2|\}$. Now Γ is a convex set since it is an open half-space. Also (16) and (17) imply $P_1 \in \Gamma$ and $P_3 \in \Gamma$. From the convexity of Γ, $P_2 \in \Gamma$. However this contradicts (15). □

Remark 2.1 Exclusion Regions associated with subsets of transmitter-receiver pairs: *Above, (6) defines the exclusion region for a* single *transmitter-receiver pair. We can generalize this concept further to* sets *of transmitter-receiver pairs. By an exclusion region $E_{T'}$ of a set T' of transmitters and their corresponding receivers, we mean a region such that for any other disjoint set of transmitter-receivers pairs T'', (i.e., $T' \cap T'' = \phi$), $E_{T'} \cap E_{T''} = \phi$.*

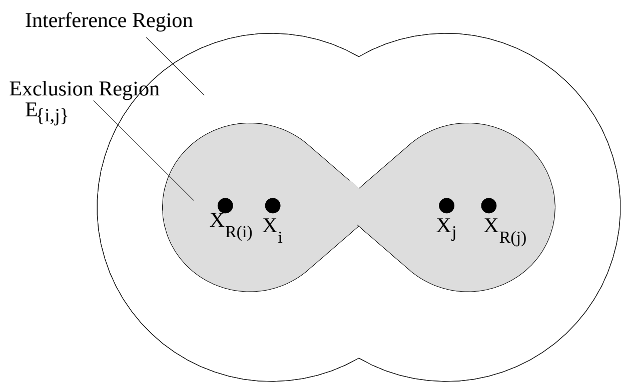

Figure 3: Exclusion Region $E_{\{i,j\}}$ for two Transmitter-Receiver Pairs.

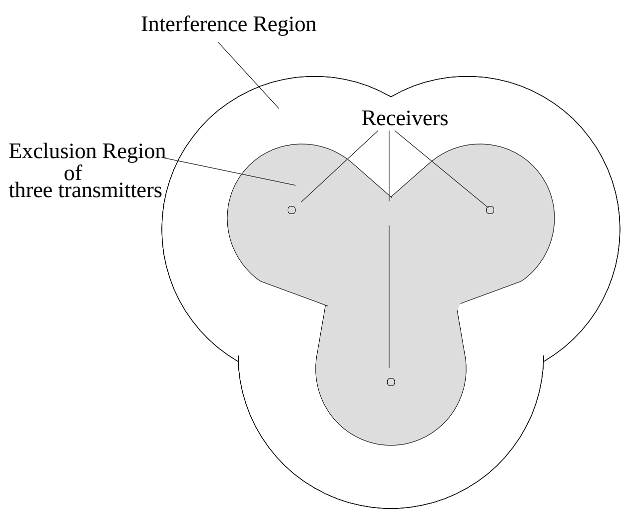

Figure 4: Exclusion Region for three Transmitter-Receiver Pairs.

Let $E(X_{R(k)}, I_k)$ denote the exclusion region, as defined in (6), of a receiver $X_{R(k)}$ with interference region I_k. Now, if we define $E_{T'}, T' \subseteq T$ as

$$E(T') = \cup_{i \in T'} E(X_{R(i)}, \cup_{k \in T'} I_k), \tag{18}$$

then it guarantees that

$$T_1, T_2 \subseteq T \ \textit{and} \ \ T_1 \cap T_2 = \phi \Rightarrow E(T_1) \cap E(T_2) = \phi. \tag{19}$$

To see this, we will show that the following holds:

$$j \notin T_1 \ \textit{and} \ \ i \notin T_2 \Rightarrow E(X_{R(i)}, \cup_{k \in T_1} I_k) \cap E(X_{R(j)}, \cup_{k \in T_2} I_k) = \phi. \tag{20}$$

This last equation can be shown by proceeding similar to the proof of Theorem 2.2 because $X_{R(i)} \notin \cup_{k \in T_2} I_k$ and $X_{R(j)} \notin \cup_{k \in T_1} I_k$. Figure 3 shows the exclusion region for two transmitter-receiver pairs under the simple Protocol Model, while Figure 4 shows the same for the case of three pairs.

3 Capacity Bounds for Wireless Networks Under the Protocol Model

We will now use the Exclusion Regions identified above in (6) to establish upper bounds for the capacity of wireless networks under the Protocol Model defined in Section 2.1.

Lemma 3.1 *If I_k is a disk of radius $(1 + \Delta)|X_k - X_{R(k)}|$ centered at X_k, then E_k is the region inside the ellipse with X_k and $X_{R(k)}$ as the foci, and eccentricity $\frac{1}{1+\Delta}$,*

$$E_k = \{P : |P - X_{R(k)}| + |P - X_k| \leq (1 + \Delta)|X_{R(k)} - X_k|\}; \tag{21}$$

see Figure 5.

Proof: As in Figure 5, let I_k represent the interference region comprising the disk of radius $(1+\Delta)|X_{R(k)} - X_k|$ centered at X_k, and let B be the Boundary of I_k. Let P be any point inside I_k, and Q be any point outside I_k. let P' and Q', respectively, be the points of intersection of $X_k P$ (extended in the direction of P away from X_k) and line segment $X_k Q$ with B. Noting that $|Q' - X_k| = |P' - X_k| = (1 + \Delta)|X_{R(k)} - X_k|$, the following holds,

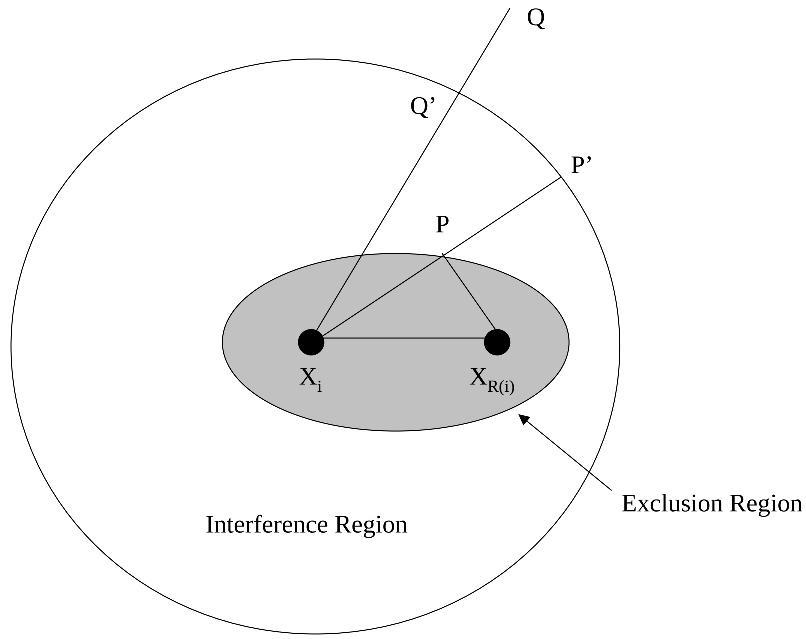

Figure 5: Exclusion Region when I_k is a disk

$$\begin{aligned} |P-Q| &> |P-Q'|, \text{ (because } \angle PQ'Q > \tfrac{\pi}{2}) \\ &\geq |Q'-X_k| - |X_k - P| \\ &= |P'-X_k| - |X_k - P| \\ &= |P'-P|. \end{aligned} \tag{22}$$

Now applying the definition of (6), we get

$$\begin{aligned} & P \in E_k \\ \Leftrightarrow\ & |P - X_{R(k)}| < |P-Q|, \forall Q \notin I_k \\ \Leftrightarrow\ & |P - X_{R(k)}| \leq |P-Q'|, \forall Q' \in B \\ \Leftrightarrow\ & |P - X_{R(k)}| \leq |P-P'|, \text{ from } (22) \\ \Leftrightarrow\ & |P - X_{R(k)}| + |P - X_k)| \leq |P' - X_k| \\ \Leftrightarrow\ & |P - X_{R(k)}| + |P - X_k)| \leq (1+\Delta)|X_{R(k)} - X_k|. \end{aligned}$$

The region given by the last expression is an ellipse with X_i and $X_{R(i)}$ as the foci, and eccentricity $\frac{1}{1+\Delta}$. □

We now consider the case where the domain of the wireless network is a disk D of unit area. Since exclusion regions associated with different transmissions are disjoint, one may regard each transmission as consuming a portion of the area within the domain. This is the key reason why area is also a resource in wireless networks (in addition to spectrum). We begin by showing how much of an exclusion region must necessarily lie inside the domain.

Lemma 3.2 *With at least two active transmitter-receiver pairs, at least a fraction $\frac{1}{4}$ of the area of the exclusion region must lie inside the domain D of the wireless network.*

Proof: Consider any transmitter-receiver pair, X_i and $X_{R(i)}$, and the (closed) region inside the ellipse, E_i. We will refer to the boundary of this region as E. Consider any other active receiver $X_{R(j)}$ at that instant. Note that all three of X_i, $X_{R(i)}$ and $X_{R(j)}$ lie inside the domain D. Hence D contains both the foci of E and at least one point on E. We show that this condition is sufficient to guarantee that D contains at least a fraction $\frac{1}{4}$ of the area of the ellipse.

Let a and b be the lengths of the major and the minor axes of E respectively. Let C be the (closed) disk that minimizes the area of $C \cap E_i$ and satisfies the constraint that it contains both the foci of E and at least one point on the boundary E. The first claim is that either C is contained inside E_i (but still touching E), or both the foci of E lie on the circumference of C. To see why, first note that for every other closed disk C' strictly contained in C, at least one of the three constraints should be unmet, else the common area would become strictly smaller. Let us suppose, for the sake of argument, that C is not contained inside E_i and at most one of the two foci of E lies on the circumference of C. We will now show that there is a C' inside C that satisfies all the three constraints. If both the foci lie strictly inside C, and C is not contained inside E_i, we can simply contract C by a small enough factor around its center. If exactly one focus, say F, lies on the circumference of C, and C is not contained in E_i, we simply contract C by moving its center in the direction of F while contracting its radius to keep F on the boundary. In both cases, we get a strictly smaller circle C' that satisfies all the three constraints, a contradiction.

So we can and do assume that either C is contained inside E_i or both the foci lie on the circumference of C. We need to show that the common area is at least $\frac{\pi ab}{16}$. For the first case, if P is a point on the boundary which is

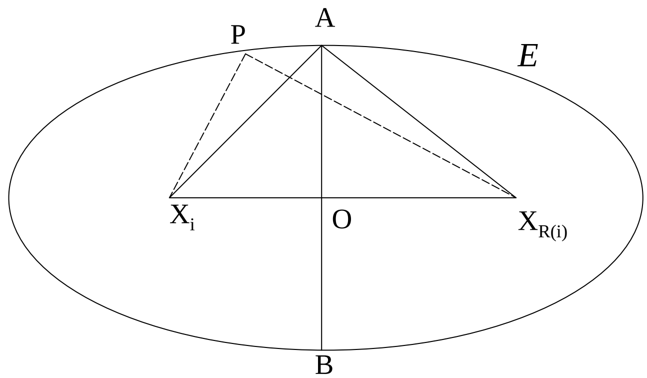

Figure 6: Ellipse E with foci X_i and $X_{R(i)}$

contained in C (see Figure 6), then the diameter of C is at least

$$\begin{aligned} max(PX_i, PX_{R(i)}) &\geq \frac{PX_i + PX_{R(i)}}{2} \\ &= \frac{a}{2}. \end{aligned}$$

Then the area of $C \cap E_i = C$ is at least $\frac{\pi a^2}{16} \geq \frac{\pi ab}{16}$.

We now consider the second case when both the foci lie on the circumference of C. This means that the center of C must lie on the perpendicular bisector of $X_i X_{R(i)}$ and we only need to consider the family, denoted by Γ, of such circles. We consider two cases – when $ae < b$ and when $ae \geq b$. These translate to $e < \frac{1}{\sqrt{2}}$ and $e \geq \frac{1}{\sqrt{2}}$ respectively.

For the first case, note that the circle with $X_i X_{R(i)}$ as diameter will not touch E since $ae < b$; see Figure 7. Hence we can assume w.l.o.g, that the center of C lies below O. Let C' be the circle in the family Γ, that is tangent to E at B, and let O' be its center. Note that C must contain the curvilinear triangle $X_i B X_{R(i)}$. We will show that this area is at least $\frac{\pi ab}{16}$. The radius, r of C' can be found using,

$$(\frac{ae}{2})^2 + (\frac{b}{2} - r)^2 = r^2.$$

Using $b^2 = a^2(1 - e^2)$, this reduces to

$$r = \frac{a^2}{4b}. \tag{23}$$

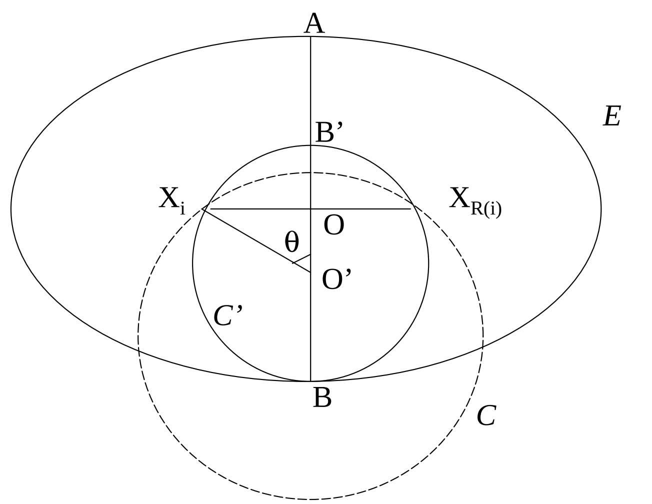

Figure 7: Ellipse E with foci X_i and $X_{R(i)}$

Next we use the fact that area of the curvilinear triangle $X_i B' X_{R(i)}$ is at most $\frac{B'O}{B'B}\pi r^2$. This is because

$$\begin{aligned}
& Area(X_i B' X_{R(i)}) \leq \frac{B'O}{B'B}\pi r'^2 \\
\Leftrightarrow\ & \theta r^2 - r^2 \sin\theta\cos\theta \leq \frac{r(1-\cos\theta)}{2r}\pi r^2 \\
\Leftrightarrow\ & \theta - \sin\theta\cos\theta \leq \frac{(1-\cos\theta)}{2}\pi \\
\Leftrightarrow\ & \frac{\pi}{2}\cos\theta - \frac{\sin(2\theta)}{2} \leq \frac{\pi}{2} - \theta \\
\Leftrightarrow\ & \frac{\pi}{2}\sin\theta' - \frac{\sin(2\theta')}{2} \leq \theta', \text{ where } \theta' = \tfrac{\pi}{2} - \theta,
\end{aligned}$$

which is true. Thus,

$$
\begin{aligned}
Area\ of\ C \cap E_i &\geq Area(X_i B X_{R(i)}) \\
&\geq \frac{BO}{B'B}\pi r^2 \\
&= \frac{b}{2}\frac{1}{2r}\pi r^2 \\
&= \frac{\pi b r}{4} \\
&= \frac{\pi a^2}{16} \text{ (using (23))} \\
&\geq \frac{\pi a b}{16},
\end{aligned}
$$

as required.

We now consider the second case, $e \geq \frac{1}{\sqrt{2}}$. This has three subcases, depending upon how the circumference of C intersects the ellipse E. The cases are: the circle intersects the ellipse at four points, the circle intersects the ellipse at two points and its center is contained inside the ellipse, and the circle cuts the ellipse at two points but its center is outside the ellipse. Note that in all these three subcases, both foci lie on the boundary of C.

For the first subcase, the circle C cuts E at four points. Then both A and B must be contained inside the circle; see Figure 8. Since both triangles, $X_i A X_{R(i)}$ and $X_i B X_{R(i)}$, are then contained inside C, the area of C is at least

$$
\begin{aligned}
& 2\frac{1}{2}(ae)\frac{b}{2} \\
\geq\ & \frac{ab}{2\sqrt{2}}, \text{ since } e \geq \frac{1}{\sqrt{2}} \\
\geq\ & \frac{\pi a b}{16}.
\end{aligned}
$$

For the second subcase, the circumference of C cuts E at two points and the center of C lies inside E_i; see Figure 8. Let the center of C be O'. Note then that C must contain B because $ae \geq b$. Let the circumference of C intersect OA at R as shown. Then we know that the area of C must be greater that the sum of areas of the triangles $X_i R X_{R(i)}$ and $X_i B X_{R(i)}$.

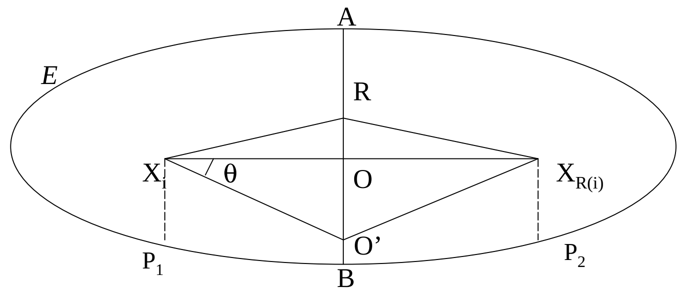

Figure 8: Ellipse E with foci X_i and $X_{R(i)}$

Note that θ, as shown in Figure 8, is smaller than $\frac{\pi}{4}$ since $X_iO \geq OB$ as assumed. Also,

$$\begin{aligned} RO &= RO' - OO' \\ &= X_iO' - OO' \\ &= \frac{ae}{2\cos\theta} - \frac{ae}{2\cot\theta} \\ &= \frac{ae\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)}{2} \\ &\geq \frac{a\frac{1}{\sqrt{2}}(\sqrt{2}-1)}{2} \quad (\text{since} \quad e \geq \frac{1}{\sqrt{2}},\ \theta \leq \frac{\pi}{4}) \\ &\geq \frac{b(\sqrt{2}-1)}{2\sqrt{2}}. \end{aligned}$$

Thus,

$$\begin{aligned} RB &= RO + OB \\ &\geq \frac{b(\sqrt{2}-1)}{2\sqrt{2}} + \frac{b}{2} \\ &= \frac{b(2\sqrt{2}-1)}{2\sqrt{2}}. \end{aligned}$$

Then the total area of triangles $X_iRX_{R(i)}$ and $X_iBX_{R(i)}$ is

$$\begin{aligned}\frac{1}{2}(ae)(RB) &\geq \frac{aeb(2\sqrt{2}-1)}{4\sqrt{2}} \\ &\geq \frac{ab(2\sqrt{2}-1)}{8} \text{ (since } e \geq \tfrac{1}{\sqrt{2}}) \\ &\geq \frac{\pi ab}{16}.\end{aligned}$$

Finally, the third subcase is where the circle intersects the ellipse at two points, but the center of the circle lies outside the ellipse. Assume that the center of C lies below the ellipse; see Figure 8. Then the two quadrilaterals, X_iP_1BO and $OBP_2X_{R(i)}$ must lie inside C. Using the equation of ellipse, $X_iP_1 = X_{R(i)}P_2 = \frac{b}{2}\sqrt{1-e^2}$. Thus the area is at least the sum of the areas of the quadilaterals, which is,

$$\begin{aligned}& 2\frac{ae}{2}\frac{1}{2}(\frac{b}{2} + \frac{b\sqrt{1-e^2}}{2}) \\ =\ & \frac{ab}{4}e(1+\sqrt{1-e^2}) \\ \geq\ & \frac{\pi ab}{16},\end{aligned}$$

where the last inequality follows because $e(1+\sqrt{1-e^2})$ attains a minimum at $e = 1$ in the interval $[\frac{1}{\sqrt{2}}, 1]$. This completes the proof. □

We now proceed to calculate the upper bounds on capacity using the exclusion region obtained above. Let there be n nodes with λ as the average throughput per node, and let $\overline{L}$ be the average distance as in Theorem 2.1 in [1]. Let the rate of the wireless channel be W bits/sec which is further subdivided into channels of rate $W_1, W_2, \cdots, W_m$ with $\sum_{i=1}^{m} W_i = W$. Assume slots of length τ for simplicity. The following result improves the upper bound on the best case capacity given by Theorem 2.1 in [1].

Theorem 3.1 *Under the Protocol Model, the best case transport capacity $\lambda n\overline{L}$ is bounded as follows:*

$$\lambda n\overline{L} \leq \sqrt{\frac{8}{\pi}}\frac{1}{\sqrt{(1+\Delta)\sqrt{\Delta}\sqrt{2+\Delta}}}W\sqrt{n}. \tag{24}$$

Proof: We proceed in a manner similar to that in [1]. Consider bit b, $1 \leq b \leq \lambda nT$. Let $h(b)$ be the number of hops taken, and let r_b^h be the distance traversed by b. Let τ be slot length, and let s be any slot. Then

$$\sum_{b=1}^{\lambda nT} \sum_{h=1}^{h(b)} r_b^h \geq \lambda nT\overline{L}. \tag{25}$$

Since in any slot, at most $n/2$ nodes can transmit, we also have

$$\sum_{b=1}^{\lambda nT} \sum_{h=1}^{h(b)} 1(\textit{The } h^{th} \textit{ hop of bit b is over subchannel m in slot s}) \leq \frac{W_m \tau n}{2}.$$

Note that there can be no more than $\frac{T}{\tau}$ slots in time T. Summing over all sub-channels and slots, we get the following

$$H \;:=\; \sum_{b=1}^{\lambda nT} h(b) \leq \frac{WTn}{2}. \tag{26}$$

We now use Lemma 3.1 to calculate the maximum number of transmissions in any slot. We need to calculate the area of the ellipse with X_k and $X_{R(k)}$ as the foci, and eccentricity $e = \frac{1}{1+\Delta}$. Note that the length of the major axis is $2a = r_k(1+\Delta)$. The length of the minor axis is given by $2b = 2a\sqrt{1-e^2} = r_k\sqrt{\Delta^2 + 2\Delta}$. The area of the ellipse is πab. Also since at least one-fourth of the ellipse will lie inside the total area, we get the exclusive area for the transmission as

$$\frac{\pi ab}{4} = \pi \frac{r_k^2}{16}(1+\Delta)\sqrt{\Delta}\sqrt{2+\Delta}. \tag{27}$$

We know that the number of bits transferred over the m^{th} channel in slot s is $W_m \tau$. This gives

$\sum_{b=1}^{\lambda nT} \sum_{h=1}^{h(b)} 1(\text{bit b's } h^{th} \text{ hop is over subchannel m in slot s})\pi \frac{{r_b^h}^2}{16}(1+\Delta)$
$\sqrt{\Delta}\sqrt{2+\Delta} \leq W_m \tau.$

Summing over the subchannels and slots, we get

$$\sum_{b=1}^{\lambda nT} \sum_{h=1}^{h(b)} \pi \frac{(r_b^h)^2}{16}(1+\Delta)\sqrt{\Delta}\sqrt{2+\Delta} \leq WT.$$

Rearranging and dividing both sides by H, we get

$$\sum_{b=1}^{\lambda nT}\sum_{h=1}^{h(b)}\frac{1}{H}(r_b^h)^2 \le \frac{16WT}{\pi H(1+\Delta)\sqrt{\Delta}\sqrt{2+\Delta}}. \tag{28}$$

Using the convexity of the square function, we have

$$(\sum_{b=1}^{\lambda nT}\sum_{h=1}^{h(b)}\frac{1}{H}(r_b^h))^2 \le \sum_{b=1}^{\lambda nT}\sum_{h=1}^{h(b)}\frac{1}{H}(r_b^h)^2. \tag{29}$$

Using (29) and (28), we get

$$\sum_{b=1}^{\lambda nT}\sum_{h=1}^{h(b)} r_b^h \le \sqrt{\frac{16HWT}{\pi(1+\Delta)\sqrt{\Delta}\sqrt{2+\Delta}}}. \tag{30}$$

Thus (25) and (30) yield

$$\lambda nT\overline{L} \le \sqrt{\frac{16WTH}{\pi(1+\Delta)\sqrt{\Delta}\sqrt{2+\Delta}}}. \tag{31}$$

Using (26) and (31) gives the required result. □

We now give a better constructive lower bound for the best case transport capacity of a wireless network under the Protocol Model than is given in [1]. The exclusion region obtained in Lemma 3.1 gives insight into the optimal arrangement of nodes for maximizing capacity. It is obvious that the best one can do is the best-case packing of ellipses in the domain. Also the convexity inequality (29) suggests that all the transmission radii should be made equal to maximize capacity. Below we see how configuring the nodes as a *grid of ellipses* gives a sufficiently good configuration so that even the dependence on Δ is of the same form as the upper bound proved above.

Theorem 3.2 *There is a placement of nodes inside the disk of unit area, and an assignment of traffic patterns, such that, under the Protocol Model, the network can achieve*

$$\sqrt{\frac{1}{\pi}}\frac{W\sqrt{n}}{\sqrt{(1+\Delta)\sqrt{\Delta}\sqrt{2+\Delta}}} \text{ bit-meters/sec.} \tag{32}$$

Proof: We will use a grid of ellipses to arrange the nodes. As in Figure 9, it is sufficient to simply stack the ellipses on top of each other, and to arrange them horizontally as shown. Let a and b be the major and minor axes lengths of the ellipse, and let e be the eccentricity. In particular, we know that $a = r(1+\Delta)$ and $e = \frac{1}{1+\Delta}$, where r is the common radius of transmission for all the pairs. The following shows how the Protocol Model constraints are met:

$$\begin{aligned} |T_1 - R_2|^2 &= (ae)^2 + b^2 \\ &= a^2e^2 + a^2(1-e^2) \\ &= a^2 = r^2(1+\Delta)^2, \end{aligned}$$

and

$$\begin{aligned} |T_1 - R_3| &= a \\ &= r(1+\Delta). \end{aligned}$$

Thus, the configuration is a feasible set of transmitter-receiver pairs as the minimum transmitter-receiver distances are maintained.

We will assume that the nodes are arranged in a square L with length $l = \sqrt{\frac{2}{\pi}}$, which is the largest square that fits into our circular domain. We will fit a regular grid of ellipses into L. The transmitter and the receivers can then be placed at the foci of the ellipses. To achieve node placement similar to that in Figure 9, in each row we will place the transmitter on the left focus for every even ellipse, and right focus of every odd ellipse.

The common radius of transmission will be chosen as

$$r = \sqrt{\frac{2lb}{n(1+\Delta)\sqrt{\Delta}\sqrt{2+\Delta}}}. \tag{33}$$

The x-separation can now be set to $a = r(1+\Delta)$ and the y-separation to $b = r\sqrt{\Delta}\sqrt{2+\Delta}$. Let n_1 and n_2 be the number of grid points in each row and column. Then, we have the following conditions:

$$\begin{aligned} l &\leq n_1 r(1+\Delta), \\ l &\leq n_2 r\sqrt{\Delta}\sqrt{2+\Delta}. \end{aligned}$$

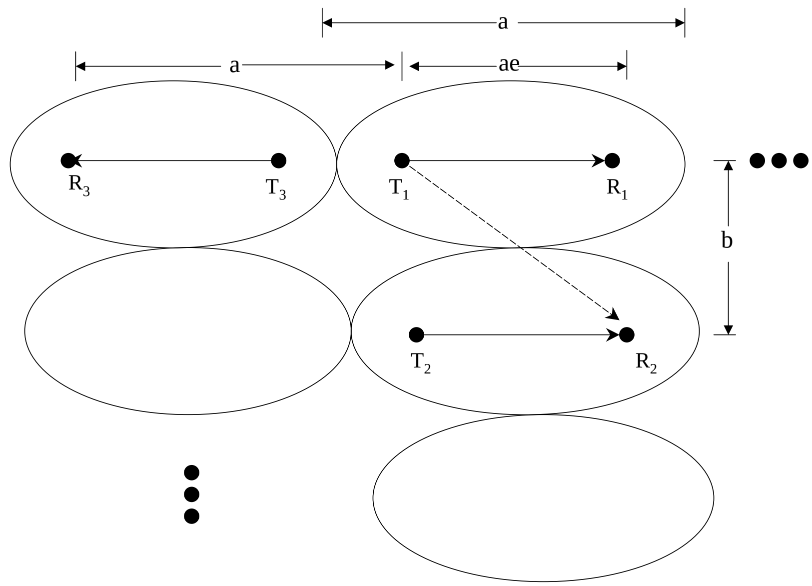

Figure 9: Sufficiency of Arranging a Grid of Ellipses

From the above, it is easy to see that the number of grid points is sufficient to place the n nodes because,

$$\begin{aligned} n_1 n_2 &\geq \frac{l^2}{r^2(1+\Delta)\sqrt{\Delta}\sqrt{2+\Delta}} \\ &= \frac{n}{2}. \end{aligned}$$

Assuming a single channel of transmission and assuming that each of the $\frac{n}{2}$ ellipses has a transmitter-receiver pair that transmits with the radius of transmission as given in (33), we obtain a transport capacity in bit-meters/sec of

$$\begin{aligned} \lambda n \overline{L} &= W \frac{n}{2} r \\ &= W \frac{n}{2} \sqrt{\frac{2l^2}{n(1+\Delta)\sqrt{\Delta}\sqrt{2+\Delta}}} \\ &= \sqrt{\frac{1}{\pi}} \frac{1}{\sqrt{(1+\Delta)\sqrt{\Delta}\sqrt{2+\Delta}}} W\sqrt{n}. \end{aligned} \tag{34}$$

This completes the proof. □

From Theorem 3.1 and Theorem 3.2, we have bracketed the best case transport capacity for wireless networks on a disk to within a factor of $\sqrt{8}$ for all Δ and n. In fact, the factor $\frac{1}{4}$ in Lemma 3.2 appears only because we have to allow large transmission ranges when n is small. Asymptotically though, these transmissions are vanishingly small, and in that case nearly all the exclusion region is in the domain. Thus, asymptotically, one can actually eliminate the factor $\frac{1}{4}$ and the best case transport capacity is bracketed to within a factor of $\sqrt{2}$, which is fairly tight.

We will now proceed to the case of random networks. We will assume as in [1] that there are n nodes scattered uniformly and randomly on the surface of a sphere S^2, or a disk D, with unit area in either case. Each nodes sends traffic to a destination chosen as the closest node to a uniformly randomly selected point in the domain. We also assume that each node uses the same range of transmission, and that the Protocol Model of communication is used. We will determine the asymptotic throughput capacity of such a network.

From [1], we know the following:

Lemma 3.3 *The asymptotic probability that graph $G(n, r(n))$ has an isolated node, and is thus disconnected, is strictly positive if $\pi r^2 = \frac{\log n + K_n}{n}$ and $\limsup_n K_n < +\infty$.*

Also, ignoring the edge effects, we get an expression similar to that in (27) for the exclusive region. This determines the maximum concurrency possible.

Lemma 3.4 *Under the Protocol Model, the number of concurrent transmissions on any particular channel can be at most $\frac{4}{c'\pi(1+\Delta)\sqrt{\Delta}\sqrt{2+\Delta}r^2(n)}$.*

We now give an upper bound for the throughput capacity of the network.

Theorem 3.3 *For random networks on S^2 or on D under the Protocol Model, there is a deterministic constant $c'' < +\infty$, not depending upon n, Δ or W, such that*

$$\lim_{n\to\infty} Prob(\lambda(n) = \frac{c''W}{(1+\Delta)\sqrt{\Delta}\sqrt{2+\Delta}\sqrt{n\log n}} \textit{ is feasible}) = 0. \qquad (35)$$

Proof: Using Lemma 3.4, and adding the transmissions over all the channels, we see that the total number of transmissions is bounded by $\frac{4W}{c'\pi(1+\Delta)\sqrt{\Delta}\sqrt{2+\Delta}r^2(n)}$ bits/sec under the Protocol Model.

Let $\overline{L}$ be the mean distance between source and destination. Since each random point has some node within an $o(1)$ radius around it, the mean length of a path is $\overline{L}-o(1)$ with high probability. Hence the number of hops is $\frac{\overline{L}-o(1)}{r(n)}$ and the total bits/sec for the entire network is at least $\frac{(\overline{L}-o(1))n\lambda(n)}{r(n)}$.
We must now have the following constraint:

$$\frac{(\overline{L}-o(1))n\lambda(n)}{r(n)} \leq \frac{4W}{c'\pi(1+\Delta)\sqrt{\Delta}\sqrt{2+\Delta}r^2(n)}.$$

Using Lemma 3.3, the above translates to

$$\lambda(n) \leq \frac{4W}{c'\sqrt{\pi}(1+\Delta)\sqrt{\Delta}\sqrt{2+\Delta}\sqrt{n\log n}(\overline{L}-o(1))}.$$

The above implies the condition stated in (35). □

4 Directional Antennas

We will now use the technique developed in Section 2 for the case of directed antennas. We first consider interference patterns arising due to dipoles. Figure 10 shows the exclusion region for the case of a center-fed half-wave dipole [4]. The shaded region is the exclusion region for the receiver shown. Figure 11 shows the exclusion region for a full-wave dipole. Next we consider interference pattern of a traveling-wave antenna. Figure 12 shows the exclusion region for the same.

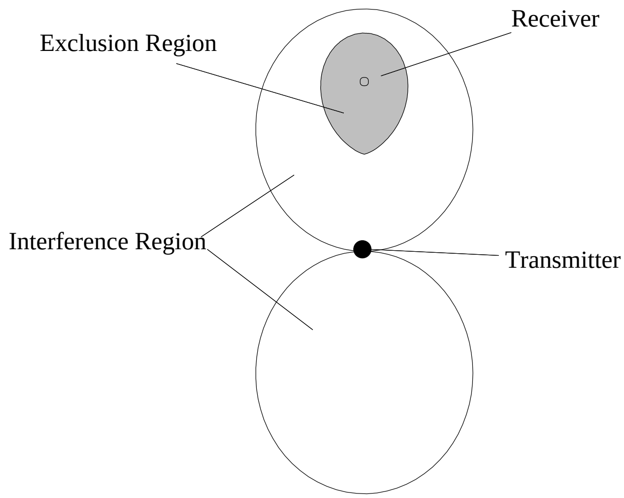

Figure 10: Exclusion Region for Center-fed Half-wave Dipole.

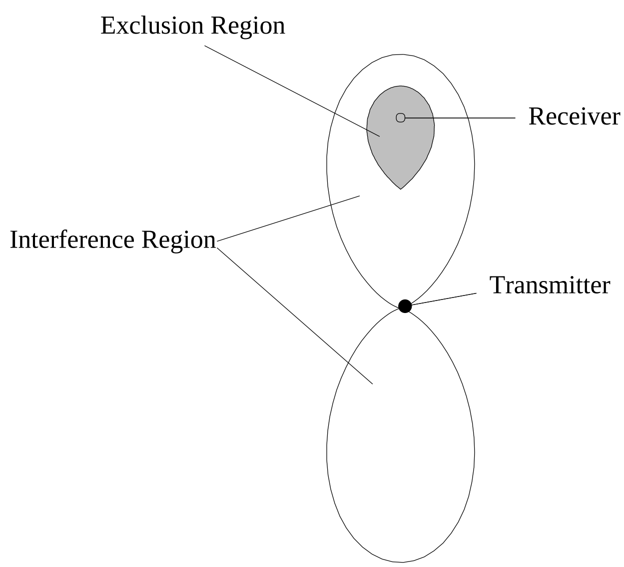

Figure 11: Exclusion Region for Center-fed Full-wave Dipole.

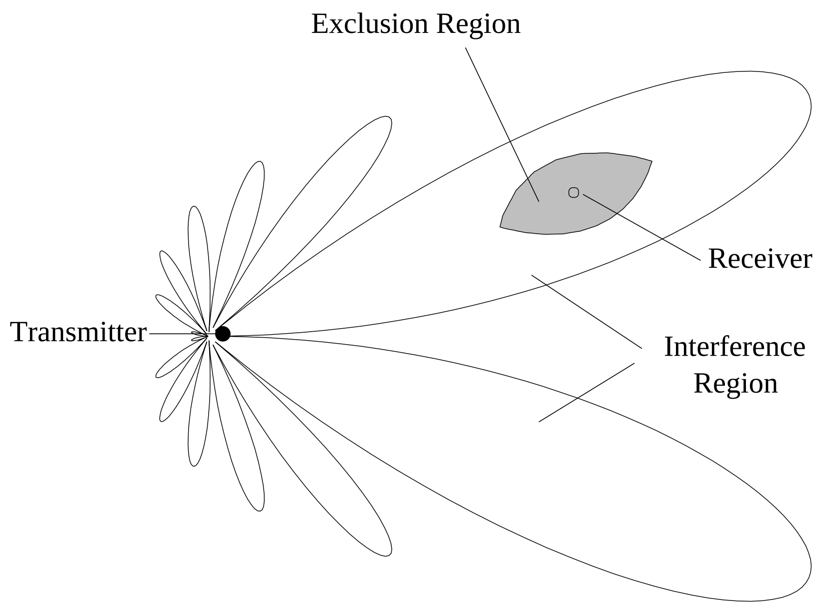

Figure 12: Exclusion Region for Traveling Wave Antenna.

5 Concluding Remarks

We have shown how the Protocol Model of [1] entails larger exclusion regions of transmissions than identified in [1]. This allows us to obtain an improved feasible transport capacity and an improved upper bound, both with the same dependence not only on n but also on Δ. The ratio of $\sqrt{8}$ between the two brackets the best case feasible transport capacity to within a factor of $\sqrt{8}$ for all values of n and Δ. Finally, the exclusion region can also be applied to the case of directional antennas.

References

[1] P. Gupta and P. R. Kumar, "The Capacity of Wireless Networks," *IEEE Transactions on Information Theory*, vol. IT-46, no. 2, pp. 388-404, March 2000.

[2] Piyush Gupta and P. R. Kumar, "Internets in the Sky: The Capacity of Three Dimensional Wireless Networks," *Communications in Information and Systems*, vol. 1, issue 1, pp. 33-49, Jan. 2001.

[3] Piyush Gupta and P. R. Kumar, "Critical Power for Asymptotic Connectivity in Wireless Networks", pp. 547-566, in *Stochastic Analysis, Control, Optimization and Applications: A Volume in Honor of W.H. Fleming. Edited by W.M. McEneany, G. Yin, and Q. Zhang, Birkhauser, Boston*, 1998. ISBN 0-8176-4078-9.

[4] E. C. Jordan and K. G. Balmain. "ElectroMagnetic Waves and Radiating Systems", Prentice-Hall, Englewood Cliffs, N.J., 1968. ISBN-81-203-0054-8.