CONVERGENCE OF ADAPTIVE CONTROL SCHEMES USING LEAST-SQUARES PARAMETER ESTIMATES [1]

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Abstract

We examine the stability, convergence, asymptotic optimality, and self-tuning properties of stochastic adaptive control schemes based on least-squares estimates of the unknown parameters, when the additive noise is i.i.d. and Gaussian, and the true system is of minimum phase. Our analysis starts with the technique of "Bayesian embedding", introduced earlier by Sternby [1], which shows that the recursive least-squares parameter estimates converge in general. Then exploiting the "normal equations" of least-squares we establish that all "stable" control law designs used in a certainty-equivalent (i.e. indirect) procedure generally yield a stable adaptive control system.

Next we obtain four results which characterize the limiting behavior precisely. The first determines the possible limits of the parameter estimates, and yields all "self-tuning" type results. The next shows that the square of a certain linear combination of outputs and inputs has average value zero, and yields all "optimality" results. The third is a similar result on exogenous inputs, and yields all results based on "persistency of excitation". The final result shows that the first few coefficients of the "A" polynomial are consistently estimated for systems with delay greater than one.

By specializing these general results we establish the convergence, asymptotic optimality and self-tuning properties of a variety of proposed adaptive control schemes, including the original self-tuning regulator with unit delay, both with and without fixing "b_0", a certainty-equivalent minimum-variance self-tuning regulator for systems with general delay, self-tuning trackers, and self-tuning pole-zero placement schemes.

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[1] The research reported here has been supported in part by the U.S. Army Research Office under Contract No. DAAL 03-88-K-0046 and in part by the Joint Services Electronics Program under Contract N00014-84-C-0149.

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As an illustrative example of the results obtained, a certainty equivalent self-tuning regulator is shown to yield strongly consistent parameter estimates when the delay is strictly greater than one, even without any excitation in the reference trajectory.
Keywords: Adaptive Control, Least-Squares Parameter Estimates, Self-tuning Controllers, Stochastic Control.

1 INTRODUCTION

Analyses of adaptive controllers for linear stochastic systems have traditionally been based on finding what is called a "stochastic Lyapunov function." This method entails a detailed study of the overall adaptive system and the establishment of various inequalities, which in turn establish the boundedness of various variables. However, it has only been successful in a few isolated instances when the parameter estimator is either a stochastic gradient algorithm, or what is called a "modified least squares algorithm", and the certainty equivalent control law is of minimum variance type; see Goodwin, Ramadge and Caines [2], Becker, Kumar and Wei [3], Kumar and Praly [4], Sin and Goodwin [5], Praly, Lin and Kumar [6]; and even then it is a very cumbersome method.

In this paper we examine the behavior of adaptive control schemes which use a recursive least-squares parameter estimator. In two important papers Sternby [1] and Rootzen and Sternby [7] have introduced a procedure of "Bayesian embedding", which shows that for systems with i.i.d. Gaussian noise the parameter estimates generally converge. Aside from this, very little is known regarding the behavior of recursive least-squares parameter estimate based adaptive control schemes. For example, whether the original minimum variance self-tuning regulator of Peterka [8] and Åström and Wittenmark [9] actually self-tunes, has been an open question for more than fifteen years. Also, no conclusive results are available for certainty equivalent control laws which are of pole-zero placement type, or based on LQG design, etc.

In this paper we extend the work of [1],[7] to analyze such recursive least-squares based adaptive control schemes, under the restrictive assumptions that the additive noise is i.i.d. and Gaussian, and the true system is of strictly minimum phase.

The starting point of our work is the result of [1], obtained by a purely probabilistic approach, which we call "Bayesian embedding", that:

i) The parameter estimates, and thus the adaptive control law, both converge a.s. (Theorem 1),

This establishes the convergence of the parameter estimates irrespective of the control law design, be it of minimum variance type, or LQG design, or of pole-zero placement type etc., and thus it has applications even outside of adaptive control. However, this wide applicability is tempered by two limitations. The additive noise entering into the system is required to be white, Gaussian noise. Second, the convergence result (i) above may not be valid on an exceptional set of true parameter vectors of Lebesgue measure zero.

Starting with this "universal convergence" result, we exploit the normal equations satisfied by the least-squares parameter estimates to obtain a variety of general and important conclusions. The first of these results is:

ii) The overall adaptively controlled system is stable in an average squared sense whenever the estimated parameters are used in a certainty-equivalent fashion to design a control law which is stable for the estimated parameters, and the true system is of minimum-phase (Theorem 3).

Thus it renders unnecessary the extremely difficult task of establishing the stability of adaptive systems, which issue has traditionally consumed the overwhelming majority of research efforts. Having established stability in general, we obtain a further set of four results which characterize precisely the asymptotic "performance" of certainty-equivalent adaptive control schemes based on least-squares parameter estimates:

iii) We characterize the possible limiting values of the parameter estimates by a certain polynomial equality, which shows that the limiting parameter estimates correctly identify the resulting closed-loop transfer function from the noise to the output (Theorem 4(i)).

When applied to specific schemes, this "closed-loop identification" result is the source of all "self-tuning" type results:

iv) We show that a certain linear combination of outputs and inputs has the average of its squared values equal to zero (Theorem 4(ii)).

This result yields all "optimality" type results for specific schemes:

v) We establish that a certain linear combination of exogenous inputs has average squared value equal to zero (Theorem 4(iii)).

This yields all results based on "persistency of excitation" of exogenous inputs. Finally, the next result is specific for systems with delay greater than one:

vi) We show that the first $d-1$ coefficients of the A-polynomial are consistently estimated for systems with delay $d > 1$ (Theorem 4(iv)).

This is useful in obtaining some strong consistency results for high delay systems.

In this paper we apply the above general results to specifically analyze the original self-tuning regulator for systems with unit delay, both with and without fixing b_0 (Sections 7 and 8), an indirect self-tuning minimum variance regulator for systems with general delay (Section 9), self-tuning trackers (Section 10), and self-tuning pole-zero placers (Section 11). For example, in Section 10 we show that an indirect self-tuning minimum variance regulator, which differs from that of Åström and Wittenmark's [9], provides strongly consistent parameter estimates whenever the delay is strictly greater than one, even though the identically zero reference trajectory of the regulation problem itself provides no excitation.

The rest of this paper is organized as follows. In Section 2, we present the method of Bayesian embedding, which shows that the least-squares parameter estimates converge. In Section 3, we examine the "normal" equations of least squares satisfied by the limiting parameter estimates. In Section 4 we describe the general adaptive control law considered, and in Section 5 we prove its stability. In Section 6, we obtain the general results characterizing the limiting behavior of the certainty-equivalent adaptive controller. Finally, in Sections 7-11, we specialize these results to the case of the original self-tuning regulator for unit delay systems, allowing for either estimating b_0 or fixing it, an indirect self-tuning minimum-variance regulator for general delay systems, self-tuning trackers, and pole-zero placement schemes.

2 CONVERGENCE OF RECURSIVE LEAST-SQUARES VIA BAYESIAN EMBEDDING

We consider the system,

$$y(t+1) = \phi^T(t)\theta^0 + w(t+1) \quad \text{for } t = 0, 1, 2, \dots \tag{1}$$

where $\theta^0 \in R^n$,

$$\phi(t) \text{ is a Borel measurable function of } (y(0), y(1), y(2), \dots, y(t)) \text{ for each } t = 0, 1, 2, \dots, \tag{2}$$

with $y(0)$ a given initial condition, and on a some probability space $(\Omega, \mathcal{F}, P)$,

$$\{w(t)\} \text{ are i.i.d. and normally distributed with } E[w(t)] = 0 \text{ and } E[w^2(t)] = \sigma^2 > 0. \tag{3}$$

The recursive least squares parameter estimator for θ^0 is defined by the following algorithm:

$$\hat{\theta}(t+1) = \hat{\theta}(t) + R^{-1}(t)\phi(t)[y(t+1) - \phi^T(t)\hat{\theta}(t)]; \quad \hat{\theta}(0) = \bar{\theta}, \tag{4}$$

$$R(t) = R(t-1) + \phi(t)\phi^T(t); \qquad R^{-1}(-1) = P_0 = P_0^T > 0. \tag{5}$$

Our goal is to analyze the convergence of the above parameter estimates for the fixed value of θ^0.

To conduct this analysis, it is mathematically convenient to presume, temporarily, that the "true" parameter θ^0 is *randomly* chosen according to a prior Gaussian distribution. We call this mathematical technique as "Bayesian embedding." The consequence of this presumption is that the recursive least-squares estimates then coincide with the conditional mean. However, this conditional mean sequence has the martingale property, and it therefore converges a.s. with respect to the white Gaussian noise sequence, and for a.e. θ^0. Since the Gaussian distribution of θ^0 is mutually absolutely continuous with respect to Lebesgue measure, we are thus able to conclude that except on a set of parameters of Lebesgue measure zero, the recursive least-squares estimates converge a.s.

This approach as well as result are due to Sternby [1]. Our only contribution is to point out that there is a need to rigorously establish that the Kalman filtering (or recursive least-squares) equations provide the conditional mean even without requiring a square-integrability assumption on $\phi(t)$, and to point out that this is done in the recent paper [10]. (Indeed, proving the square-integrability of $\phi(t)$ is comparable in difficulty with establishing the stability of such an adaptive system). For the sake of completeness we provide a proof.

Theorem 1 *Consider the system (1, 2, 3) with the recursive least squares parameter estimator (4, 5). Then there exists a set $N \subseteq R^n$ with $l(N) = 0$, where l denotes Lebesgue measure, such that if*

$$\theta^0 \notin N, \tag{6}$$

then,

$$\lim_{t\to\infty} \hat{\theta}(t) = \hat{\theta}(\infty) \quad P - a.s.,$$

where $\hat{\theta}(\infty)$ is a P-a.s. finite random vector.

Proof: Without loss of generality we will assume that $\Omega = R^\infty, \mathcal{F} = \mathcal{B}^\infty$, where $\mathcal{B} =$ Borel $\sigma-$algebra and that $w(t,\omega) := \omega_t$ for $\omega = (\omega_1, \omega_1, \omega_2, \ldots) \in$

R^∞. (see Theorem 3, p. 189 of Chow and Teicher [11]). Also let $(R^n, \mathcal{B}^n, \mu)$ be a probability space such that if $\theta^0(\omega') := \omega'$ for $\omega' \in R^n$, then $\theta^0 \sim N(\bar{\theta}, \frac{1}{\sigma^2}P_0)$. Now let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}) := (R^n \times R^\infty, \mathcal{B}^n \times \mathcal{B}^\infty, \mu \times P)$ be the product probability space. Note that θ^0 and $\{w(t)\}$ are independent, and that $y(t), \phi(t), \hat{\theta}(t)$, and $R(t)$ are all random variables on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$. (We refer to this process of studying the behavior of the system for a fixed θ^0, by studying its behavior when θ^0 is a random variable with prior distribution $N(\bar{\theta}, \frac{1}{\sigma^2}P_0)$, as *Bayesian embedding*).

Let $\mathcal{Y}_t := \sigma(y(0), y(1), \ldots, y(t))$ be the sub σ - algebra of $\tilde{\mathcal{F}}$ generated by $(y(0), y(1), \ldots, y(t))$. It is shown in Chen, Kumar and van Schuppen [10] that (2) and (3) result in,

$$\hat{\theta}(t) = E[\theta^0 | \mathcal{Y}_t] \quad \tilde{P} - a.s. \tag{7}$$

This is a statement that the Kalman filtering equations (4,5) yield the conditional mean of the parameter vector under Gaussianity and independence assumptions on $\{w(t)\}$ and θ^0, and (2). The central contribution of [10] is that it shows that square-integrability of $\phi(t)$ is *not* needed for the validity of (7). Thus it extends the results in Liptser and Shiryayev [12] which require such a square-integrability; see assumptions (1-4) on page 62 and Theorem 13.4, or Example 1 on page 85, of [12].

Since θ^0 has finite moments of all orders, $\{\hat{\theta}(t), \mathcal{Y}_t\}$ is a convergent martingale, i.e.,

$$\lim_{t \to \infty} \hat{\theta}(t, \tilde{\omega}) = \theta^*(\tilde{\omega}) \quad \tilde{P} \text{ - a.e. } \tilde{\omega},$$

where θ^* is $\tilde{P}$ -a.s. finite, see Doob [13]. Thus there exists a set $\tilde{A} \subseteq \tilde{\Omega}$, with $\tilde{P}(\tilde{A}) = 1$, such that

$$\lim_{t \to \infty} \hat{\theta}(t, \tilde{\omega}) = \theta^*(\tilde{\omega}) \text{ for all } \tilde{\omega} \in \tilde{A}.$$

Thus

$$\int_{\tilde{\Omega}} 1_{\tilde{A}}(\tilde{\omega}) d\tilde{P}(\tilde{\omega}) = 1,$$

where $1_{\tilde{A}}$ is the indicator function of $\tilde{A}$, i.e. $1_{\tilde{A}}(\tilde{\omega}) = 0$ if $\tilde{\omega} \in \tilde{A}^c$, and $1_{\tilde{A}}(\tilde{\omega}) = 1$ if $\tilde{\omega} \in \tilde{A}$. By Fubini's Theorem, see Theorem 2, p. 180 of Chow and Teicher [11],

$$1 = \int_{\tilde{\Omega}} 1_{\tilde{A}}(\tilde{\omega}) d\tilde{P}(\tilde{\omega}) = \int_{R^n} \left[\int_{R^\infty} 1_{\tilde{A}}(\omega', \omega) dP(\omega) \right] d\mu(\omega')$$

Hence $\int_{R^\infty} 1_{\tilde{A}}(\omega',\omega)dP(\omega) = 1 \quad \mu - a.e.\ \omega' \in R^n$. Let $N \subseteq R^n$, be the set with $\mu(N) = 0$ such that

$$\int_{R^\infty} 1_{\tilde{A}}(\omega',\omega)dP(\omega) = 1 \quad \text{ for all } \omega' \in N^c.$$

Since Lebesgue measure on R^n is mutually absolutely continuous with respect to μ, the Gaussian measure, we have

$$l(N) = 0.$$

Also noting that $\theta^0(\omega') = \omega'$, we thus have

$$P(\{\omega \in \Omega| \quad \lim_{t\to\infty} \hat{\theta}(t,\theta^0,\omega) \text{ exists and is finite }\}) = 1 \text{ for all } \theta^0 \in N^c.$$

This concludes the proof. □.

Thus we have proved the key result that except when the true parameter vector lies in a certain exceptional set of true parameter vectors N of Lebesgue measure zero, the recursive least squares algorithm yields convergent parameter estimates (though not necessarily to the true parameter vector!). Moreover, this convergence occurs irrespective of the control law design used. Even for other simpler algorithms such as the stochastic gradient algorithm, the convergence of the parameter estimates has been demonstrated only in isolated cases such as [3], [4].

The serious limitations of the above result are that; (i) the noise sequence $\{w(t)\}$ is required to be white and Gaussian, which is a significant restriction; and (ii) the result may not hold for θ^0 in a certain null set (with respect to Lebesgue measure). We hope that at least the restriction of Gaussianity of $w(t)$ in (i), and quite possibly the restriction (ii) too, can be removed, under appropriate moment conditions on $w(t)$.

Throughout the rest of this paper we will assume (6).

3 THE NORMAL EQUATIONS FOR THE LIMITING PARAMETER ESTIMATE

Having established the convergence of the parameter estimates $\hat{\theta}(t)$ to some (random) $\hat{\theta}(\infty)$ in the previous section, it remains to determine what $\hat{\theta}(\infty)$ is. The key to doing this, as well as to establishing the stability of adaptive control systems and their asymptotic behavior, is the following result which relies on the facts that the parameter estimates converge and that they satisfy the "normal" equations of least squares.

Theorem 2 *Consider the system (1,2,3,4,5, 6). Let* $r(N-1) := \sum_{t=0}^{N-1} \phi^T(t)\phi(t)$. *Then,*

$$\lim \frac{1}{r(N-1)} \sum_{t=0}^{N-1} \phi(t)\phi^T(t)(\hat{\theta}(t) - \theta^0) \quad = 0 \quad a.s..$$

Proof: From (4,5) we have $R(t)\hat{\theta}(t+1) = R(t)\hat{\theta}(t) + \phi(t)[y(t+1) - \phi^T(t)\hat{\theta}(t)] = R(t-1)\hat{\theta}(t) + \phi(t)y(t+1)$. Summing, we obtain $R(N-1)\hat{\theta}(N) - R(-1)\hat{\theta}(0) = \sum_{t=0}^{N-1} \phi(t)y(t+1)$. Since $R(N-1) = R(-1) + \sum_{t=0}^{N-1} \phi(t)\phi^T(t)$, $R(-1) = P_0^{-1}$, and $\hat{\theta}(0) = \bar{\theta}$, it follows that the parameter estimates satisfy the following "normal" equations,

$$\left[\sum_{t=0}^{N-1} \phi(t)\phi^T(t)\right] \hat{\theta}(N) = \left[\sum_{t=0}^{N-1} \phi(t)y(t+1)\right] + P_0^{-1}[\bar{\theta} - \hat{\theta}(N)]. \qquad (8)$$

Now note that due to (2, 3), we have $\liminf_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N-1} y^2(t) \geq \sigma^2 \quad a.s.$; see Lemma 4 of Kumar and Praly [4] for full details. As a particular consequence we have, $r(N-1) \geq \sum_{t=0}^{N-1} y^2(t) \to +\infty \quad a.s.$. From the local convergence Theorem for martingales; see Chow [14], we then obtain, $\lim \frac{1}{r(N-1)} \sum_{t=0}^{N-1} \phi(t)w(t+1) = 0$ a.s.. Hence using $y(t+1) = \phi^T(t)\theta^0 + w(t+1)$, we have

$$\frac{1}{r(N-1)} \sum_{t=0}^{N-1} \phi(t)y(t+1) = \frac{1}{r(N-1)} \sum_{t=0}^{N-1} \phi(t)\phi^T(t)\theta^0 + o(1) \text{ a.s..}$$

Thus,

$$\frac{\text{RHS of (8)}}{r(N-1)} = \frac{1}{r(N-1)} \sum_{t=0}^{N-1} \phi(t)\phi^T(t)\theta^0 + o(1) \text{ a.s.,} \qquad (9)$$

since $P_0^{-1}[\bar{\theta} - \hat{\theta}(N)]$ is bounded (in fact, convergent).

Now turn to the LHS of (8), and note that since $\{\hat{\theta}(t)\}$ is a Cauchy sequence, i.e. $(\hat{\theta}(N) - \hat{\theta}(t)) \to 0$ as $t, N \to +\infty$, we have

$$\begin{aligned}\frac{\text{LHS of (8)}}{r(N-1)} &= \frac{1}{r(N-1)} \sum_{t=0}^{N-1} \phi(t)\phi^T(t)\hat{\theta}(t) + \frac{1}{r(N-1)} \sum_{t=0}^{N-1} \phi(t)\phi^T(t)[\hat{\theta}(N) - \hat{\theta}(t)] \\ &= \frac{1}{r(N-1)} \sum_{t=0}^{N-1} \phi(t)\phi^T(t)\hat{\theta}(t) + o(1) \quad \text{a.s..}\end{aligned}$$

Equating this with (9) proves the Theorem. □

It should be noted that Rootzen and Sternby [7] have shown a related result that $\sum_{t=0}^{\infty}[\phi^T(t)(\hat{\theta}(\infty)-\theta^0)]^2 < \infty \quad a.s.$; see Theorem 5 of [7].

It is worth noting that till now we have made no assumptions on $\phi(t)$, save that it be finite a.s., and $\mathcal{Y}_t$-measurable.

4 THE ADAPTIVE CONTROL SYSTEM

We now turn to the specific case where (1) is the linear stochastic system,

$$A(q^{-1})y(t) = q^{-d}B(q^{-1})u(t) + w(t), \text{ with } \quad A(q^{-1}) \neq 1 \text{ and } B(q^{-1}) \neq 0, \tag{10}$$

where q^{-1} is the backward shift operator, $A(q^{-1}) := 1-\sum_{i=0}^{p} a_i q^{-i-1}$ and $B(q^{-1}) := \sum_{i=0}^{s} b_i q^{-i}$. The integer $d \geq 1$ is the *delay.* Note that in the notation of (1), $\phi(t) := (y(t), y(t-1), \ldots, y(t-p), u(t-d+1), u(t-d), \ldots, u(t-d-s+1))^T$ and $\theta^0 := (a_0, a_1, \ldots, a_p, b_0, b_1, \ldots, b_s)^T$. Correspondingly, we shall denote the entries of $\hat{\theta}(t)$ by $\hat{\theta}(t) =: (\hat{a}_0(t), \ldots, \hat{a}_p(t), \hat{b}_0(t), \ldots, \hat{b}_s(t))^T$.

We introduce the notation $A(\theta; q^{-1}) := 1-\sum_{i=0}^{p} \theta_i q^{-i-1}$ and $B(\theta; q^{-1}) := \sum_{i=0}^{s} \theta_{p+i+1} q^{-i}$, where $(\theta_0, \ldots, \theta_p, \theta_{p+1}, \ldots, \theta_{p+s+1})^T := \theta$. Thus note that $A(\hat{\theta}(t); q^{-1})$ and $B(\hat{\theta}(t); q^{-1})$ are the "estimates" of $A(q^{-1})$ and $B(q^{-1})$.

We shall consider a general adaptive control law for $u(t)$ which is of the form:

$$\sum_{i=0}^{n} r_i(\hat{\theta}(t))u(t-i) = \sum_{i=0}^{m} s_i(\hat{\theta}(t))y(t-i) + \sum_{i=0}^{l} t_i(\hat{\theta}(t))z^*(t-i) \quad a.s..$$

A variety of "indirect", i.e. certainty-equivalent, adaptive control laws are of this form, as we shall see in the sequel. For convenience, we shall write the above adaptive control law in the form,

$$R(\hat{\theta}(t); q^{-1})u(t) = S(\hat{\theta}(t); q^{-1})y(t) + T(\hat{\theta}(t); q^{-1})z^*(t) \quad a.s. \tag{11}$$

where, $R(\hat{\theta}(t); q^{-1}) = \sum_{i=0}^{n} r_i(\hat{\theta}(t))q^{-i}$, etc, with the convention that in the time-varying polynomials $R(\hat{\theta}(t))$, $S(\hat{\theta}(t))$ and $T(\hat{\theta}(t))$, the shift operator q^{-1} is assumed to act only on time dependent variables to the *right* of it.

We shall make the following assumptions concerning the control law (11).

For every $\theta \epsilon R^{p+s+2}$, there exists $R'(\theta; q^{-1}) = \sum_{i=0}^{n-s} r'_i(\theta)q^{-i}$, a polynomial in q^{-1}, with $r'_0(\theta) = 1$, such that $R(\theta; q^{-1}) = R'(\theta; q^{-1})B(\theta; q^{-1})$, (12)

$$\text{The coefficients } r'_i, s_i \text{ and } t_i \text{ are continuous in } \theta \epsilon R^{p+s+2}, \tag{13}$$

All the roots of the polynomial $H(\theta;q^{-1}) := A(\theta;q^{-1})R'(\theta;q^{-1}) - q^{-d}S(\theta;q^{-1})$ are inside the open unit disk $\{z : |z| < 1\}$, (14)

$$\{z^*(t)\} \text{ is a bounded, deterministic (i.e. exogenous) sequence.} \qquad (15)$$

The motivation for this control law as well as its implications are as follows. For the system with parameter θ,

$$A(\theta;q^{-1})y(t) = q^{-d}B(\theta;q^{-1})u(t) + w(t),$$

if the control law

$$R(\theta;q^{-1})u(t) = S(\theta;q^{-1})y(t) + T(\theta;q^{-1})z^*(t)$$

is applied, then the resulting closed-loop system is

$$y(t) = \frac{q^{-d}T(\theta;q^{-1})}{H(\theta;q^{-1})}z^*(t) + \frac{R'(\theta;q^{-1})}{H(\theta;q^{-1})}w(t).$$

Thus, the above control law allows the zeroes of the system to be cancelled, as well as its poles to be moved. This will necessitate the following minimum phase assumption on the *true* system:

$$\text{All the roots of } B(q^{-1}) \text{ are inside the open unit disk,} \qquad (16)$$

as we shall see in what follows. Also, because the closed loop poles are given by those of $H(\theta;q^{-1})$, it is very natural to require that the control law be designed so that it at least stabilizes the *estimated* system, and so we make the assumption (14) above. Finally, allowing the exogenous signal $z^*(t)$ in the control law allows us to incorporate reference trajectories or command signals in the adaptive control law.

5 THE STABILITY OF INDIRECT ADAPTIVE CONTROL LAWS

We now show the general result that any adaptive control law of the form described above will result in a stable system.

Theorem 3 *Consider the adaptive control system (10,16,3,6,4,5,11,12,13,14,15). Then,*

i) $\limsup_{N\to+\infty} \frac{1}{N}\sum_{t=0}^{N-1} y^2(t) \quad < \quad +\infty \quad a.s.,$

ii) $\limsup_{N\to+\infty}\frac{1}{N}\sum_{t=0}^{N-1}u^2(t) \quad < \quad +\infty \quad a.s..$

Proof: Note first that by the minimum phase assumption (16) and (3),

$$\begin{aligned}\sum_{t=0}^{N-1} y^2(t) \leq r(N-1) &= 0(\sum_{t=0}^{N-1} y^2(t)) + 0(\sum_{t=0}^{N-1} u^2(t-d+1)) + 0(N) \quad a.s. \\ &= 0(\sum_{t=0}^{N} y^2(t)) \quad a.s., \end{aligned} \tag{17}$$

see [2]. Hence, from Theorem 2, we have

$$\frac{1}{\sum_{t=0}^{N} y^2(t)} \sum_{t=0}^{N-1} [\phi^T(t)(\hat{\theta}(t)-\theta^0)]^2 \to 0 \quad a.s. \tag{18}$$

Substituting $\phi^T(t)\theta^0 = y(t+1) - w(t+1)$, and

$$\phi^T(t)\hat{\theta}(t) = [1 - A(\hat{\theta}(t); q^{-1})]y(t+1) + B(\hat{\theta}(t); q^{-1})q^{-d}u(t+1), \quad a.s. \tag{19}$$

in (18) gives,

$$\frac{1}{\sum_{t=0}^{N} y^2(t)} \sum_{t=0}^{N-1} [A(\hat{\theta}(t); q^{-1})y(t+1) - B(\hat{\theta}(t); q^{-1})q^{-d}u(t+1) - w(t+1)]^2 \quad \to 0 \quad a.s..$$

Now note that the coefficients of $R'(\hat{\theta}(t-d+1); q^{-1})$ are bounded due to the convergence of $\hat{\theta}(t)$ and the continuity of $r_i'(\theta)$. Hence using the Schwarz inequality we can multiply on the left by $R'(\hat{\theta}(t-d+1); q^{-1})$ inside the summation above to get,

$$\begin{aligned}\frac{1}{\sum_{t=0}^{N} y^2(t)} \sum_{t=0}^{N-1} [R'(\hat{\theta}(t-d+1); q^{-1})A(\hat{\theta}(t); q^{-1})y(t+1) - R'(\hat{\theta}(t-d+1); q^{-1})B(\hat{\theta}(t); q^{-1})q^{-d}u(t+1) \\ -R'(\hat{\theta}(t-d+1); q^{-1})w(t+1)]^2 \to 0 \quad a.s.. \end{aligned} \tag{20}$$

Since $\|\hat{\theta}(t) - \hat{\theta}(t-1)\| \to 0$, $\sum_{t=0}^{N-1} u^2(t+1-d) = 0(\sum_{t=0}^{N} y^2(t))$ by the minimum phase assumption (16), and due to (13), we can commute the polynomials B and R' to get

$$\sum_{t=0}^{N-1} [q^{-d}R(\hat{\theta}(t-d+1); q^{-1})u(t+1) - B(\hat{\theta}(t); q^{-1})R'(\hat{\theta}(t-d+1); q^{-1})q^{-d}u(t+1)]^2 = o(\sum_{t=0}^{N} y^2(t)) \quad a.s.$$

Thus we can replace $R'(\hat{\theta}(t-d+1);q^{-1})B(\hat{\theta}(t);q^{-1})q^{-d}u(t+1)$ by $R(\hat{\theta}(t-d+1);q^{-1})u(t-d+1)$ in (20) to get,

$$\frac{1}{\sum_{t=0}^{N} y^2(t)} \sum_{t=0}^{N-1} [R'(\hat{\theta}(t-d+1);q^{-1})A(\hat{\theta}(t);q^{-1})y(t+1)-R(\hat{\theta}(t-d+1);q^{-1})u(t-d+1)$$
$$-R'(\hat{\theta}(t-d+1);q^{-1})w(t+1)]^2 \to 0 \quad a.s.$$

Now substituting the control law (11), and using $\hat{\theta}(t) \to \hat{\theta}(\infty)$, (13), and the definition (14), we obtain,

$$\frac{1}{\sum_{t=0}^{N} y^2(t)} \sum_{t=0}^{N-1} [H(\hat{\theta}(\infty);q^{-1})y(t+1)-q^{-d}T(\hat{\theta}(\infty);q^{-1})z^*(t+1)-R'(\hat{\theta}(\infty);q^{-1})w(t+1)]^2$$
$$\to 0 \quad a.s. \tag{21}$$

Let us now define $\delta(t)$ by

$$\delta(t+1) := H(\hat{\theta}(\infty);q^{-1})y(t+1)-q^{-d}T(\hat{\theta}(\infty);q^{-1})z^*(t+1)-R'(\hat{\theta}(\infty);q^{-1})w(t+1).$$

Note that y can be regarded as the output of a stable linear system (since $H(\hat{\theta}(\infty);q^{-1})$ has all its roots inside the unit disk by virtue of (14)) driven by δ, Tz^* and $R'w$. Hence,

$$\sum_{t=0}^{N-1} y^2(t+1) = 0(\sum_{t=0}^{N-1} z^{*2}(t+1-d))+0(\sum_{t=0}^{N-1} w^2(t+1))+0(\sum_{t=0}^{N-1} \delta^2(t+1)) \quad a.s..$$

Since (21) shows that $\sum_{t=0}^{N-1} \delta^2(t+1) = o(\sum_{t=0}^{N} y^2(t))$, we obtain

$$\begin{aligned}\sum_{t=0}^{N-1} y^2(t+1) &= 0(\sum_{t=0}^{N-1} z^{*2}(t+1-d) + 0(\sum_{t=0}^{N-1} w^2(t+1)) \\ &= 0(N) \quad a.s., \end{aligned} \tag{22}$$

where we have used the facts that $z^*(t)$ is bounded, and $\frac{1}{N}\sum_{t=0}^{N-1} w^2(t+1) \to \sigma^2$ a.s. due to (3). This proves the mean square stability results (i, ii). □

6 CHARACTERIZATION OF ASYMPTOTIC PERFORMANCE OF INDIRECT ADAPTIVE CONTROL LAWS

In this section we characterize precisely the asymptotic "performance" of the general certainty-equivalent adaptive control law of Section 4.

The first result (i) of the following Theorem shows that the limiting parameter estimates correctly estimate the resulting *closed-loop* transfer function from the noise to the output. It is the source of all "self-tuning" type results for various adaptive control schemes. The second result (ii), showing that the average of the square of a certain linear combination of inputs and outputs is zero along sample paths, is the source of all "optimality" conclusions. The third is a similar result about the exogenous inputs, and it yields all conclusions based on "persistency of excitation" of exogenous inputs. (See for example Theorem 8(iii) or Theorem 9(ii) of the sequel, where sufficient richness of $\{z^*(t)\}$ leads to the conclusion that $\hat{\theta}(\infty) = \theta^0$). Finally, (iv) is valid only for systems with delay $d \geq 2$, and it shows that the first $d-1$ parameters $a_0, a_1, \ldots, a_{d-2}$ are consistently estimated. It is useful in proving some strong consistency results.

Theorem 4 *Consider the adaptive control system (10,16,3,6,4,5,11,12,13,14,15). Then,*

i) $[A(\hat{\theta}(\infty); q^{-1}) - A(q^{-1})]R(\hat{\theta}(\infty); q^{-1}) = q^{-d}[B(\hat{\theta}(\infty); q^{-1}) - B(q^{-1})]S(\hat{\theta}(\infty); q^{-1})$ *a.s.*

ii) $\lim_{N\to+\infty} \frac{1}{N}\sum_{t=0}^{N-1}[H(\hat{\theta}(\infty); q^{-1})y(t) - q^{-d}T(\hat{\theta}(\infty); q^{-1})z^*(t) - R'(\hat{\theta}(\infty); q^{-1})w(t)]^2 = 0 \quad a.s.$

iii) $\lim_{N\to+\infty} \frac{1}{N}\sum_{t=0}^{N-1}[(B(\hat{\theta}(\infty), q^{-1}) - B(q^{-1}))T(\hat{\theta}(\infty); q^{-1})z^*(t)]^2 = 0 \quad a.s.$

iv) *If* $d \geq 2$, *then* $\hat{a}_i(\infty) = a_i$ *for* $0 \leq i \leq d-2 \quad a.s.$

Proof: The result (ii) is a direct consequence of (21,22). To prove the remaining results, note that due to (17) and (22), the result of Theorem 2 yields $\frac{1}{N}\sum_{t=0}^{N-1}[\phi^T(t)(\hat{\theta}(t) - \theta^0)]^2 \to 0 \quad a.s.$. Substituting (19) and $\phi^T(t)\theta^0 = [1 - A(q^{-1})]y(t+1) + B(q^{-1})q^{-d}u(t+1)$ gives,

$$\frac{1}{N}\sum_{t=0}^{N-1}[(A(\hat{\theta}(t); q^{-1}) - A(q^{-1}))y(t+1) + (B(q^{-1}) - B(\hat{\theta}(t); q^{-1}))q^{-d}u(t+1)]^2 \to 0 \quad a.s..$$

Now the result (iv) follows just as in the establishment of (21.i) of Lemma 14 in [3]. Moreover, as in the proof of Theorem 3, we can multiply by $R(\hat{\theta}(t); q^{-1})$ and substitute for the control law (11) to obtain,

$$\frac{1}{N}\sum_{t=0}^{N-1}[((A(\hat{\theta}(t); q^{-1}) - A(q^{-1}))R(\hat{\theta}(t); q^{-1}) - q^{-d}(B(\hat{\theta}(t); q^{-1}) - B(q^{-1}))S(\hat{\theta}(t); q^{-1}))y(t+1)$$

$$+ q^{-d}(B(q^{-1}) - B(\hat{\theta}(t); q^{-1}))T(\hat{\theta}(t); q^{-1})z^*(t+1)]^2 \to 0 \quad a.s..$$

The results (i) and (iii) then follow by repeated application of Lemma 11 of [3]. □

In the remaining sections, we apply the general results of Theorems 3 and 4 to a variety of adaptive control schemes and obtain a number of convergence, asymptotic optimality, and self-tuning results.

7 THE ASTROM-WITTENMARK SELF-TUNING REGULATOR: UNIT DELAY CASE

Consider the unit delay, i.e. $d = 1$, system

$$A(q^{-1})y(t) = q^{-1}B(q^{-1})u(t) + w(t), \quad A(q^{-1}) \neq 1 \text{ and } B(q^{-1}) \neq 0. \quad (23)$$

The self-tuning regulator of Åström and Wittenmark [4] employs least squares estimates of the parameters in a certainty-equivalent fashion to minimize the variance of y.

The resulting adaptive control law is,

$$u(t) = -\frac{1}{\hat{b}_0(t)}\left[\sum_{i=0}^{p}\hat{a}_i(t)y(t-i) + \sum_{i=1}^{p}\hat{b}_i(t)u(t-i)\right] \quad a.s.. \quad (24)$$

It may be noted that Meyn and Caines [15] have shown that $\hat{b}_0 = 0$ is a zero probability event, and so the control law is well defined a.s.. (Note the a.s. qualifier in (11)). The adaptive control law can therefore be written as $B(\hat{\theta}(t); q^{-1})u(t) = [A(\hat{\theta}(t); q^{-1}) - 1]qy(t)$ a.s.. Hence we make the associations, $R'(\theta; q^{-1}) := 1$, $S(\theta; q^{-1}) := [A(\theta; q^{-1}) - 1]q$, $H(\theta; q^{-1}) := 1$, while $T(\theta; q^{-1}) := 0$ and $z^*(t) := 0$.

From Theorem 4(ii), we immediately get, $\lim \frac{1}{N}\sum_{t=0}^{N-1}(y(t) - w(t))^2 = 0$ *a.s.*, which establishes the *optimality* of the self-tuning regulator vis-a-vis the minimum variance cost criterion. Also note that Theorem 4(i) immediately yields,

$$\left[A(q^{-1}) - 1\right]B(\hat{\theta}(\infty); q^{-1}) = \left[A(\hat{\theta}(\infty); q^{-1}) - 1\right]B(q^{-1}). \quad a.s.. \quad (25)$$

However, $B(\hat{\theta}(\infty); q^{-1}) = A(\hat{\theta}(\infty); q^{-1}) - 1 = 0$ is a possible solution corresponding to $\hat{\theta}(\infty) = 0$. If $\hat{\theta}(\infty) \neq 0$, then (25) shows that $\frac{B(\hat{\theta}(\infty); q^{-1})}{A(\hat{\theta}(\infty); q^{-1}) - 1} =$

$\frac{B(q^{-1})}{A(q^{-1})-1}$. Hence the adaptive controller converges to an optimal (i.e. minimum variance) control law, i.e. the adaptive controller possesses the *self-tuning* property. Finally, note that if $B(q^{-1})$ and $(A(q^{-1})-1)$ have no common factors, and in addition either deg $A(q^{-1}) = p+1$ or deg $B(q^{-1}) = s$, then (25) shows that $A(\hat{\theta}(\infty); q^{-1}) = \gamma A(q^{-1})$ and $B(\hat{\theta}(\infty); q^{-1}) = \gamma B(q^{-1})$, i.e. $\hat{\theta}(\infty) = \gamma\theta^0$ for some scalar γ. We summarize these results in the following Theorem.

Theorem 5 *Consider the adaptive control system (23,16,3,6,4,5,24). Then,*

i) $\lim \frac{1}{N}\sum_{t=0}^{N-1}(y(t) - w(t))^2 = 0 \quad a.s.$

ii) *If* $\hat{\theta}(\infty) \neq 0$, *then* $\frac{B(\hat{\theta}(\infty);q^{-1})}{A(\hat{\theta}(\infty);q^{-1})-1} = \frac{B(q^{-1})}{A(q^{-1})-1}$ *a.s.. Hence the adaptive controller can be said to "self-tune" to a minimum variance regulator.*

iii) *If* $B(q^{-1})$ *and* $(A(q^{-1})-1)$ *have no common factors, and in addition either deg* $A(q^{-1}) = p+1$ *(i.e.* $a_p \neq 0$*) or deg* $B(q^{-1}) = s$ *(i.e.* $b_s \neq 0$*), then* $\hat{\theta}(\infty) = \gamma\theta^0$ *a.s. where* γ *is a scalar random variable. Hence the parameter estimates converge to a random multiple of the true parameters.*

8 SELF-TUNING REGULATOR WITH FIXED b_0

As Theorem 5(iii) shows, there is a one degree of freedom identifiability problem in the self-tuning regulator with unit delay. If $b_0 \neq 0$ is known, then there is no need to estimate it, as suggested by Åström and Wittenmark [4]. We now show that fixing b_0 eliminates this identifiability problem.

Let us rewrite the system (23) as $y(t+1) - b_0u(t) = \phi^T(t)\theta^0 + w(t+1)$, where

$$\phi^T(t) := (y(t), y(t-1), \ldots, y(t-p), u(t-1), \ldots, u(t-s))^T \text{ and } \theta^0 := (a_0, \ldots, a_p, b_1, \ldots, b_s)^T. \tag{26}$$

Then the recursive least squares estimator for θ^0 is given by,

$$\hat{\theta}(t+1) = \hat{\theta}(t) + R^{-1}(t)\phi(t)\left[y(t+1) - b_0u(t) - \phi^T(t)\hat{\theta}(t)\right]; \hat{\theta}(0) = \bar{\theta} \tag{27}$$

with $R(t)$ given as before by (5). Denoting the entries of $\hat{\theta}(t)$ by $\hat{\theta}(t) =: (\hat{a}_0(t), \ldots, \hat{a}_p(t), \hat{b}_1(t), \ldots, \hat{b}_s(t))^T$, the certainty equivalent minimum-variance control law is,

$$u(t) = -\frac{1}{b_0}\left[\sum_{i=0}^{p}\hat{a}_i(t)y(t-i) + \sum_{i=1}^{s}\hat{b}_i(t)u(t-i).\right] \quad (28)$$

There is a slight difference between (27) and (4) due to the presence of the term $b_0u(t)$ in the RHS of (27), and so the results of Theorem 4 as well as the arguments leading up to it are slightly modified. First, Theorem 1 is still valid, and so the parameters converge as before. The normal equations are modified to, $\left[\sum_{t=0}^{N-1}\phi(t)\phi^T(t)\right]\hat{\theta}(N) = \left[\sum_{t=0}^{N-1}\phi(t)[y(t+1) - b_0u(t)]\right] + P_0^{-1}[\bar{\theta}-\hat{\theta}(N)]$. The results of Theorem 2 are still valid, and so premultiplying by $(\hat{\theta}(t) - \theta^0)^T$ gives $\frac{1}{r(N-1)}\sum_{t=0}^{N-1}[\phi^T(t)((\hat{\theta}(t) - \theta^0)]^2 \to 0$ a.s.. Substituting $\phi^T(t)\hat{\theta}(t) = -b_0u(t)$, which follows from (28), gives $\frac{1}{r(N-1)}\sum_{t=0}^{N-1}[\phi^T(t)\theta^0 + b_0u(t)]^2 \to 0$. By virtue of the minimum phase assumption (16), once again we obtain the optimality result of Theorem 5(i). Regarding the limiting parameter estimates, since $b_0 \neq 0$ is fixed, arguments similar to those of Theorem 4(i) result in

$$\frac{\sum_{i=0}^{p}\hat{a}_i(\infty)q^{-i}}{b_0 + \sum_{i=1}^{s}\hat{b}_i(\infty)q^{-i}} = \frac{\sum_{i=0}^{p}a_iq^{-i}}{\sum_{i=0}^{s}b_iq^{-i}} \quad a.s. \quad (29)$$

Thus one has self-tuning. If moreover $(A(q^{-1}) - 1)$ and $B(q^{-1})$ have no common factors, and in addition either deg $A(q^{-1}) = p+1$ or deg $B(q^{-1}) = s$, then since $b_0 \neq 0$ is fixed, we obtain $\hat{\theta}(\infty) = \theta^0 \quad a.s.$. These results are summarized in the following Theorem.

Theorem 6 *Consider the adaptive control system (23,16,3,6,26,27,5,28) with $b_0 \neq 0$. Then,*

i) *The adaptive controller yields $\lim \frac{1}{N}\sum_{t=0}^{N-1}(y(t) - w(t))^2 = 0$ a.s., i.e. it is optimal vis-a-vis the minimum variance cost criterion.*

ii) *It self-tunes a.s., i.e. (29) holds.*

iii) *If $B(q^{-1})$ and $(A(q^{-1}) - 1)$ have no common factors, and in addition either deg $A(q^{-1}) = p+1$ or deg $B(q^{-1}) = s$, then $\hat{\theta}(\infty) = \theta^0 \quad a.s.$, i.e. we have strong consistency of the parameter estimates.*

9 A SELF-TUNING MINIMUM-VARIANCE REGULATOR FOR THE GENERAL DELAY CASE

We now consider the system (10) for the case of general delay $d > 1$. Let

$$R'(\theta; q^{-1}) = \sum_{i=0}^{d-1} r_i'(\theta) q^{-i}, \text{ and } S(\theta; q^{-1}) = \sum_{i=0}^{p} s_i(\theta) q^{-i} \tag{30}$$

satisfy,

$$A(\theta; q^{-1}) R'(\theta; q^{-1}) - q^{-d} S(\theta; q^{-1}) = 1 \tag{31}$$

Note that R' and S are obtained by dividing 1 by A. Both R' and S have coefficients which are continuous in θ. For θ^0 we shall denote $R'(\theta^0; q^{-1})$ and $S(\theta^0; q^{-1})$ by $R'(q^{-1})$ and $S(q^{-1})$ respectively.

The minimum variance control law for the system with parameter vector θ is then given by (see Åström [16]),

$$B(\theta; q^{-1}) R'(\theta; q^{-1}) u(t) = S(\theta; q^{-1}) y(t).$$

The resulting certainty equivalent control adaptive control law is thus of the form in (11,12) with $T(\theta; q^{-1}) = 0$. Note from (14) that $H(\theta; q^{-1}) = 1$.

Theorem 7 *Consider the adaptive control system (10,16,3,6,4,5,11,12,30,31), with $T(\theta; q^{-1}) = 0$. Then,*

i) $\lim \frac{1}{N} \sum_{t=0}^{N-1} \left[y(t) - R'(q^{-1}) w(t) \right]^2 = 0$ *a.s., i.e. the adaptive controller is optimal with respect to the minimum variance cost criterion.*

ii) $R'(\hat{\theta}(\infty); q^{-1}) = R'(q^{-1})$; $B(\hat{\theta}(\infty); q^{-1}) S(q^{-1}) = B(q^{-1}) S(\hat{\theta}(\infty); q^{-1})$ *and* $\hat{a}_0(\infty) = a_0, \hat{a}_1(\infty) = a_1, \ldots, \hat{a}_{d-2}(\infty) = a_{d-2}$ *a.s..*

iii) *If* $(a_0, a_1, \ldots, a_{d-2}) \neq (0, 0 \ldots, 0)$, *then* $\frac{S(\hat{\theta}(\infty); q^{-1})}{B(\hat{\theta}(\infty); q^{-1}) R'(\hat{\theta}(\infty); q^{-1})} = \frac{S(q^{-1})}{B(q^{-1}) R'(q^{-1})}$ *a.s. and the adaptive controller thus self-tunes a.s. to a minimum variance regulator.*

iv) *If* $(a_0, a_1, \ldots, a_{d-2}) \neq (0, 0 \ldots, 0)$, $B(q^{-1})$ *and* $S(q^{-1})$ *have no common factors, and deg* $B(q^{-1}) = s$ *(i.e.* $b_s \neq 0$*), then* $\hat{\theta}(\infty) = \theta^0$, *i.e. the parameter estimates are strongly consistent.*

v) *If* $B(q^{-1})$ *and* $S(q^{-1})$ *have no common factors, and deg* $B(q^{-1}) = s$, *but* $(a_0, \ldots, a_{d-2}) = (0, \ldots, 0)$, *then* $\hat{\theta}(\infty) = \gamma \theta^0$ *a.s. where* γ *is a scalar random variable.*

Proof: From Theorem 4(iv), $A(\hat{\theta}(\infty); q^{-1}) - A(q^{-1})$ is divisible by q^{-d}. Hence, $A(\hat{\theta}(\infty); q^{-1})(R'(q^{-1}) - R'(\hat{\theta}(\infty); q^{-1})) = (A(q^{-1})R'(q^{-1}) - 1) - (A(\hat{\theta}(\infty); q^{-1})R'(\hat{\theta}(\infty); q^{-1}) - 1) + (A(\hat{\theta}(\infty); q^{-1}) - A(q^{-1}))R'(q^{-1})$ is also divisible by q^{-d}, since each of the terms on the RHS is so divisible. Since the constant term of $A(\theta; q^{-1})$ is 1, it follows that q^{-d} divides $R'(q^{-1}) - R'(\hat{\theta}(\infty); q^{-1})$. However since $R'(q^{-1})$ and $R'(\hat{\theta}(\infty); q^{-1})$ are of degree $(d-1)$, this implies $R'(q^{-1}) - R'(\hat{\theta}(\infty); q^{-1}) = 0$, proving the first equality in (ii). Now note that from Theorem 4(ii) we immediately obtain (i). Moreover, using (31) and (12) in Theorem 4(i) gives $B(\hat{\theta}(\infty); q^{-1})\left[1 - A(q^{-1})R'(\hat{\theta}(\infty); q^{-1})\right] = -q^{-d}B(q^{-1})S(\hat{\theta}(\infty); q^{-1})$. *a.s.*. Using $R'(\hat{\theta}(\infty); q^{-1}) = R'(q^{-1})$ and (31) gives the second equality in (ii). Theorem 4(iv) directly yields the last set of equalities in (ii), thus completing the proof of (ii). To show (iii) we only need to show that $B(\hat{\theta}(\infty); q^{-1}) \neq 0$. This is done by contradiction. If $B(\hat{\theta}(\infty); q^{-1}) = 0$ then from (ii), $S(\hat{\theta}(\infty); q^{-1}) = 0$ which shows through (31) that $A(\hat{\theta}(\infty); q^{-1})R'(\hat{\theta}(\infty); q^{-1}) = 1$. This implies that $A(\hat{\theta}(\infty); q^{-1}) = 1$, which in turn implies from (ii) that $(a_0, \ldots, a_{d-2}) = (0, \ldots, 0)$, a contradiction. To prove (iv), from the second equality of (ii) it follows that $B(\hat{\theta}(\infty); q^{-1}) = \gamma B(q^{-1})$ and $S(\hat{\theta}(\infty); q^{-1}) = \gamma S(q^{-1})$ for some scalar $\gamma \neq 0$. But then from (i) and (31),

$$
\begin{aligned}
\left[\gamma A(q^{-1}) - A(\hat{\theta}(\infty); q^{-1})\right] R'(q^{-1}) &= \gamma A(q^{-1})R'(q^{-1}) - A(\hat{\theta}(\infty); q^{-1})R'(\hat{\theta}(\infty); q^{-1}) \\
&= \gamma[1 + q^{-d}S(q^{-1})] - [1 + q^{-d}S(\hat{\theta}(\infty); q^{-1})] \quad a.s.. \\
&= \gamma - 1. \qquad (32)
\end{aligned}
$$

If $\gamma = 1$, then $A(q^{-1}) = A(\hat{\theta}(\infty); q^{-1})$ since the constant term of $R'(q^{-1})$ is 1, and so we obtain the desired result. On the other hand, if $\gamma \neq 1$, then $\gamma A(q^{-1}) - A(\hat{\theta}(\infty); q^{-1}) = \gamma - 1$ and $R'(q^{-1}) = 1$ and, as before, the latter contradicts $(a_0, \ldots, a_{d-2}) \neq (0, \ldots, 0)$. For (v), note that $B(\hat{\theta}(\infty); q^{-1}) = 0$ implies $S(\hat{\theta}(\infty); q^{-1}) = 0$ which in turn implies $A(\hat{\theta}(\infty); q^{-1}) = 1$, and so $\hat{\theta}(\infty) = 0$, i.e. $\gamma = 0$. If however $B(\hat{\theta}(\infty); q^{-1}) \neq 0$, then $B(\hat{\theta}(\infty); q^{-1}) = \gamma B(q^{-1})$and $S(\hat{\theta}(\infty); q^{-1}) = \gamma S(q^{-1})$ for some scalar γ. But then repeating the argument of (32) shows $\hat{\theta}(\infty) = \gamma\theta^0$. □

The result of Theorem 7(iv) is worth highlighting. It shows that the parameter estimates are strongly consistent whenever the delay d is greater than one, even though there is no excitation in the identically zero reference trajectory of the regulation problem.

It should be noted that the adaptive controller defined above for $d > 1$ differs from that suggested by Åström and Wittenmark [9]. They suggest rewriting the system (10) as

$$y(t) = q^{-d}\left[-S(q^{-1})y(t) + B(q^{-1})R'(q^{-1})u(t)\right] + R'(q^{-1})w(t),$$

which the reader can verify by multiplying both sides of (10) by $R'(q^{-1})$ and then using the identity (31). Then they "directly" attempt to estimate the coefficients of $S(q^{-1})$ and $B(q^{-1})R'(q^{-1})$. The scheme studied here is, in contrast, a truly *indirect* scheme. One disadvantage with their scheme is that it corresponds to the case (v) above, and so it has an identifiability problem. Hence a result such as Theorem 7(iv) may not hold. However, the significant advantage of their approach is that it also works (a conjecture only, as noted in Section 1) for the case where $w(t)$ is a moving average of white noise, subject to a positive real assumption; see Ljung [17], Goodwin, Ramadge and Caines [2] and Becker, Kumar and Wei [3]. We should note that, with the exception that a stochastic gradient parameter estimation algorithm is used, the same indirect adaptive control scheme as here has been studied earlier in Fuchs [18], where its optimality is established.

10 SELF-TUNING MINIMUM-VARIANCE TRACKERS

Suppose that the goal of control is to ensure that the output $y(t)$ follows a given bounded reference trajectory $y^*(t)$ as closely as possible. This can be cast as a problem of minimizing the variance of the tracking error $(y(t) - y^*(t))$. We shall treat the cases $d = 1$ and $d > 1$ together.

The control law which minimizes the variance of the tracking error for a system with parameter θ is,

$$B(\theta; q^{-1})R'(\theta; q^{-1})u(t) = S(\theta; q^{-1})y(t) + y^*(t+d) \tag{33}$$

where $R'(\theta; q^{-1})$ and $S(\theta; q^{-1})$ are as in Section 9. Note that when $d = 1$, $R'(\theta; q^{-1}) = 1$ and $S(\theta; q^{-1}) = q[A(\theta; q^{-1}) - 1]$. In general, for $d > 1$, one needs to solve (31) for $R'(\theta; q^{-1})$ and $S(\theta; q^{-1})$.

The resulting certainty-equivalent adaptive control law is given by (11) with

$$T(\theta; q^{-1}) = 1, \text{ and } \quad z^*(t) = y^*(t+d), \text{ a deterministic bounded sequence.} \tag{34}$$

Theorem 8 *Consider the adaptive control system (10,16,3,6,4,5,11,12,30,31,34). Then,*

i) $\lim \frac{1}{N} \sum_{t=0}^{N-1} [y(t) - y^*(t) - R'(q^{-1})w(t)]^2 = 0$ *a.s., i.e. the adaptive controller is optimal vis-a-vis the variance of the tracking error.*

ii) *When $d = 1$, the results of Theorem 5(ii,iii) hold (except that one may not use the label of "self-tuning", unless $B(\hat{\theta}(\infty); q^{-1}) = B(q^{-1})$, i.e., $\gamma = 1$).*

iii) *When $d = 1$, if in addition to the conditions of Theorem 5(iii),* $\limsup \frac{1}{N} \sum_{t=0}^{N-1} [B(q^{-1})y^*(t)]^2 > 0$, *then $\hat{\theta}(\infty) = \theta^0$, and so the parameter estimates are strongly consistent. In particular this is guaranteed if $y^*(t)$ is sufficiently rich of order $(s+1)$, i.e.,* $\limsup \frac{1}{N} \sum_{t=0}^{N-1} [L(q^{-1})y^*(t)]^2 > 0$ *for all non-zero polynomials $L(q^{-1})$ of degree less than or equal to s.*

iv) *When $d > 1$, the results of Theorem 7(ii,iii,iv,v) hold (except that one may not apply the label "self-tuning" unless $\hat{\theta}(\infty) = \theta^0$).*

v) *If in addition to the conditions of Theorem 7(v), we have* $\limsup \frac{1}{N} \sum_{t=0}^{N-1} [B(q^{-1})y^*(t)]^2 > 0$, *then again, $\hat{\theta}(\infty) = \theta^0$ a.s., and the parameter estimates are strongly consistent. This is guaranteed if $y^*(t)$ is sufficiently rich of order $(s+1)$.*

Proof: The results (ii) and (iv) are immediate. In either case, since $R'(\hat{\theta}(\infty); q^{-1}) = R'(q^{-1})$ a.s., the result (i) follows from Theorem 4(ii). For (iii) and (v), note that from Theorem 4(iii), $\lim \frac{1}{N} \sum_{t=0}^{N-1} [(B(\hat{\theta}(\infty); q^{-1}) - B(q^{-1}))y^*(t)]^2 = 0$ a.s. Hence (ii,iv) yield $(\gamma - 1)^2 \lim \frac{1}{N} \sum_{t=0}^{N-1} [B(q^{-1})y^*(t)]^2 = 0$ a.s., which in turn shows that $\gamma = 1$ a.s. □

As in Section 9, it is worth noting that strong consistency of the parameter estimates can be obtained even without any excitation in the reference trajectory. These results may be compared with those of Kumar and Praly [4].

11 SELF-TUNING POLE-ZERO PLACERS

Let $u^*(t)$ be a bounded deterministic command input, and suppose that the goal of control is to make the closed-loop transfer function from $u^*(t)$ to $y(t)$ equal to $\frac{q^{-d} B_m(q^{-1})}{A_m(q^{-1})}$, where $A_m(q^{-1})$ is a polynomial with constant term equal to 1, and with all roots in the open unit disk. This is the so-called "servo-problem".

For a system with parameter θ, this goal can be achieved by choosing the control law,

$$B(\theta;q^{-1})R'(\theta;q^{-1})u(t) = S(\theta;q^{-1})y(t) + B_m(q^{-1})A_0(q^{-1})u^*(t), \qquad (35)$$

where $R'(\theta;q^{-1})$ and $S(\theta;q^{-1})$ satisfy the Diophantine equation,

$$A(\theta;q^{-1})R'(\theta;q^{-1}) - q^{-d}S(\theta;q^{-1}) = A_m(q^{-1})A_0(q^{-1}), \text{ and} \qquad (36)$$

$A_0(q^{-1})$ is any polynomial with constant term 1, and with roots inside the open unit disk; (37)

see Åström and Wittenmark [19]. Note that this results in the closed loop system, $y(t) = q^{-d}\frac{B_m(q^{-1})A_0(q^{-1})}{A_m(q^{-1})A_0(q^{-1})}u^*(t) + \frac{R'(\theta;q^{-1})}{A_m(q^{-1})A_0(q^{-1})}w(t)$, and hence produces the desired response from command input $u^*(t)$ to output $y(t)$. The zeroes of $A_0(q^{-1})$ can be regarded as the poles of the "observer" part of the design.

However, there is also some freedom to shape the response from the *noise* $w(t)$ to the output $y(t)$, even while constraining the response from the command input $u^*(t)$ to output. This freedom arises from the fact that there are infinitely many solutions $R'(\theta;q^{-1})$ and $S(\theta;q^{-1})$ to the Diophantine equation (36); see Kucera [20]. In particular, if $R'(\theta;q^{-1})$ and $S(\theta;q^{-1})$ are any arbitrary solutions of (36), then $\bar{R}'(\theta;q^{-1}) = R'(\theta;q^{-1}) + q^{-d}\lambda(q^{-1})$, and $\bar{S}(\theta;q^{-1}) = S(\theta;q^{-1}) + A(\theta;q^{-1})\lambda(q^{-1})$ are also solutions of (36) for *any* choice of $\lambda(q^{-1})$. In fact, this is the set of all solutions. As noted in [19], the corresponding transfer function from $w(t)$ to $y(t)$ is $\frac{R'(\theta;q^{-1})+q^{-d}\lambda(q^{-1})}{A_m(q^{-1})A_0(q^{-1})}$. Thus, by choice of $\lambda(q^{-1})$ (and also $A_0(q^{-1})$, if so desired), the response to the noise can be shaped.

We will however, for brevity, consider only one specific solution, the so-called "minimum degree solution" for $R'(\theta;q^{-1})$. It is obtained by constraining the degrees of R' and S in the Diophantine equation (36) to satisfy,

$$\deg R'(\theta;q^{-1}) \le d-1, \qquad (38)$$

$$\deg S(\theta;q^{-1}) \le \deg A_m(q^{-1}) + \deg A_0(q^{-1}) - d. \qquad (39)$$

The solution is unique, has the minimal degree for $R'(\theta;q^{-1})$ among all the solutions of (36); see Kucera [20], and can be obtained by simply dividing the polynomial $A_m(q^{-1})A_0(q^{-1})$ by $A(\theta;q^{-1})$. It should be noted that the coefficients of R' and S' are continuous in θ.

The resulting certainty-equivalent adaptive control law is then given by (11,12), with

$$T(\theta; q^{-1}) := B_m(q^{-1})A_0(q^{-1}), \text{ and } z^*(t) := u^*(t). \tag{40}$$

The following Theorem analyzes its performance.

Theorem 9 *Consider the adaptive control system (10,16,3,6,4,5,11,12,36,37,38,39,40). Then,*

i) $\lim \frac{1}{N}\sum_{t=0}^{N-1}[A_m(q^{-1})A_0(q^{-1})y(t) - q^{-d}B_m(q^{-1})A_0(q^{-1})u^*(t) - R'(\hat{\theta}(\infty); q^{-1})w(t)]^2 = 0$ *a.s.. In the particular sense implied by this equation, one may say that the desired response from the command input $u^*(t)$ to $y(t)$ has been met.*

ii) *If $B_m(q^{-1})A_0(q^{-1})u^*(t)$ is sufficiently rich of order $(s+1)$, then $\hat{\theta}(\infty) = \theta^0$, i.e. the parameter estimates are strongly consistent.*

Proof: The result (i) is an immediate consequence of Theorem 4(ii). For (ii), simply note that from Theorem 4(iii), we have $B(\hat{\theta}(\infty); q^{-1}) = B(q^{-1})$ a.s. by the richness conditions on $T(q^{-1})u^*(t)$. From Theorem 4(i), one then has $A(\hat{\theta}(\infty); q^{-1}) = A(q^{-1})$ by noting that since $A_m(q^{-1})$ and $A_0(q^{-1})$ have constant coefficient equal to one, $R'(\hat{\theta}(\infty); q^{-1}) \neq 0$. □

12 CONCLUDING REMARKS

For minimum phase systems where the additive noise is i.i.d. and Gaussian, the method of Bayesian embedding, and the analysis of the normal equations of least-squares, allow us to obtain the general results of Theorems 3 and 4 for a variety of adaptive control schemes using least squares estimates of the parameters.

It would be of considerable interest to remove at least the Gaussianity restriction on the distribution of $w(t)$. The "whiteness" restriction cannot be removed so easily. When it is not satisfied, one is essentially faced with a "partially observed" nonlinear system. It would also be of considerable interest to show that the exceptional set of Theorem 1 is really an empty set.

We believe that there is a clear need for a "broad spectrum" theory which allows conclusions about a wide variety of adaptive control schemes.

The present paper is a start in this direction for minimum phase systems under the restrictive assumption of an i.i.d. Gaussian noise.

Acknowledgements

The idea for this paper grew out of a discussion with Peter Caines on the trek to Loewenburg and Drachenfels at the Fourth Bad Honnef Conference on Stochastic Differential Systems in June 1988. The author is also grateful to Han-Fu Chen, Lei Guo, T. L. Lai, Sean Meyn, Sanjeev Naik and Laurent Praly for subsequent discussions. The author worked on part of this paper while he was visiting the Australian National University in July-August 1988, and he is grateful to Brian Anderson for his warm hospitality.

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