ROBUST CONTINUOUS TIME ADAPTIVE CONTROL BY PARAMETER PROJECTION *

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Abstract

We consider the problem of adaptive control of a continuous time plant of arbitrary relative degree, in the presence of bounded disturbances as well as unmodeled dynamics. The adaptation law we consider is the usual gradient update law with parameter projection, the latter being the only robustness enhancement modification employed. We show that if the unmodeled dynamics, which consists of multiplicative as well as additive system uncertainty, is small enough, then all the signals in the closed loop system are bounded. This shows that extra modifications such as, for example, normalization by a specially constructed signal, or relative dead zones, are not necessary for robustness with respect to bounded disturbances and small unmodeled dynamics. In the nominal case, where unmodeled dynamics and disturbances are absent, the asymptotic error in tracking a given reference signal is zero. Moreover, the performance of the adaptive controller is also robust in that the mean-square tracking error is quadratic in the magnitude of the unmodeled dynamics and bounded disturbances, when both are present.

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1 Introduction

Recently, there have been many attempts to study the adaptive control of plants with bounded disturbances and unmodeled dynamics. In his pioneering work, Egardt [5] showed that even small bounded disturbances can cause instability in adaptively controlled plants. He further demonstrated that modification of the adaptation law by projecting the parameter estimates, at each time instant, onto a compact, convex set known to contain the true parameter vector, however provides stability with respect to bounded disturbances.

Proving the stability of an adaptive control system in the presence of unmodeled dynamics is however more difficult, and researchers have proposed various additional modifications to the adaptation law to analyze and bound the effects of unmodeled dynamics. For example, in [8] [9] [10], Praly introduced the device of using a normalizing signal in the parameter estimator, while other modifications have included σ-modification [12] [14], normalized dead-zone [11], etc. An excellent unification of these various modifications and results can be found in Tao and Ioannou [18].

In this paper, we show that Egardt's original simple modification of parameter projection is in fact enough to provide stability with respect to small unmodeled dynamics as well as bounded disturbances. Our main results are the following:

(i) A certainty equivalent adaptive controller, using a gradient based parameter estimator with projection, ensures that all closed loop signals are bounded, when applied to a nominally minimum phase continuous time system with bounded disturbances and small unmodeled dynamics (Theorem 9.1).

(ii) In the absence of unmodeled dynamics and disturbances, i.e. in the nominal case, the error in tracking a reference trajectory converges to zero (Theorem 10.1). When unmodeled dynamics as well as bounded disturbances are present, the mean-squared tracking error is quadratic in the magnitude of the unmodeled dynamics and bounded disturbances (Theorem 11.1). Thus the adaptive controller provides robust performance in addition to robust boundedness.

While our work thus shows that a simple modification is sufficient to ensure robust boundedness and robust performance, and some early modifications may have been proposed due to the limitations of proof techniques used, we feel that it is nevertheless important for future work to compare the various modifications on the basis of the amount of robustness provided, the resulting performance as a function of unmodeled effects, transient response, etc., as well as the complexities of the modifications themselves.

The key stimulus for our work here is the recent paper of Ydstie [19], which showed that parameter projection in a gradient update law is sufficient for ensuring the boundedness of closed loop signals for a nominally minimum phase, unit delay, discrete time plant with some types of unmodeled dynamics as well as bounded disturbances.

The continuous time systems studied here give rise to several additional issues such as filtering of signals, parametrization of systems, differentiability considerations of signals, augmented errors, etc., which motivate various changes, and allow us to establish stability for nominal plants with arbitrary positive relative degree, as well as for a class of unmodeled dynamics which is larger than those considered earlier, for example in [21],[14] or [19]. For instance, unlike Ydstie[19] we do not require the true plant to be stably invertible; only the nominal plant is assumed to be minimum-phase. Additionally, in contrast to [14] we also allow the unmodeled dynamics to be nonlinear or time-varying, and do not require differentiability of either the bounded disturbance, which is lumped together with the unmodeled dynamics in our treatment, or the reference input.

The rest of the paper is organized as follows: Section 2 introduces the system and reference models. In Section 3, we reparametrize these models, and describe, in Section 4, the adaptive control law. Our analysis starts in Section 5, where we show that all closed-loop signals in the system are bounded by a particular signal $m(t)$. In Section 6, we introduce a signal $z(t)$ defined through a "switched system" which overbounds $m(t)$. In Section 7, we show that the filtered signals are comparable to z over certain bounded intervals of time. To apply these results to the stability analysis of the closed-loop system, in Section 8 we present a nonminimal representation of the closed-loop system, which is then used in Section 9 to complete the boundedness analysis by showing that a certain positive definite function of the signal $z(t)$ and the non-minimal system state error $e(t)$ is bounded. In Section 10, we show that asymptotic tracking is achieved in the nominal case, and in Section 11 we establish a mean-square robust performance result. In Section 12 we present simulation examples to illustrate the results. Finally, Section 13 presents some concluding remarks. Some necessary technical results are collected in Appendices A and B. A preliminary version of the results presented here is contained in [21].

2 System and Reference Models

Consider the single-input, single-output system,

$$y(t) = \frac{B(s)}{A(s)}(1 + \mu_m \Delta_m(s))u(t) + v(t), \tag{1}$$

where $\frac{B(s)}{A(s)}$ is the transfer function of the *modeled* part of the plant, $\Delta_m(s)$ represents the multiplicative uncertainty in the plant, and $v(t)$ represents the effect of additional additive unmodeled dynamics as well as bounded disturbances.

We will make the following assumptions on the nominal model of the plant:

(A1) $A(s) = s^n + \sum_{j=0}^{n-1} a_j s^j$, and $B(s) = \sum_{j=0}^{m} b_j s^j$, $0 \leq m < n$.

(A2) $B(s-p_0)$ is Hurwitz for some $p_0 > 0$, and $b_m \geq b_{min} > 0$. We will denote by $n_r := n-m$, the relative degree of the nominal plant.

We will make the following assumptions on the *unmodeled dynamics* and *bounded disturbance* of the plant:

(A3) The multiplicative uncertainty, $\Delta_m(s)$, is a transfer function with relative degree greater than or equal to $(1 - n_r)$ such that $\Delta_m(s - p_0)$ is stable.

(A4) The additive unmodeled dynamics and disturbances give rise to a signal $v(t)$ which satisfies $|v(t)| \leq K_v m(t) + k_v + k_{v0} exp[-pt]$, where $m(\cdot)$ is defined by,

$$\dot{m}(t) = -d_0 m(t) + d_1(|u(t)| + |y(t)| + 1), \quad m(0) \geq \tfrac{d_1}{d_0}, \tag{2}$$

All the constants are positive. Furthermore $0 < p \leq d_0$, and $d_0 + d_2 < p_0 < 2d_0 + d_2$, for some $d_2 > 0$. The term $k_{v0} exp[-pt]$ above allows for the effect of initial conditions.

Example. *A class of unmodeled dynamics, possibly nonlinear, satisfying (A4) is,*

$$v(t) = f(t, g_1(t) \cdot H_1(s)y(t), g_2(t) \cdot H_2(s)u(t))$$

where f is any affine function or any non-linear function satisfying $|f(t, x_1(t), x_2(t)| \leq k_1|x_1(t)| + k_2|x_2(t)| + k_3$, $g_1(t), g_2(t)$ are bounded time-varying signals, and $H_1(s), H_2(s)$ are strictly proper transfer functions such that $H_1(s-p_0)$ and $H_2(s-p_0)$ are stable.

The goal of adaptation is to follow the output of a reference model given by

$$y_m(t) = W_m(s)r(t) \tag{3}$$

where $W_m(s)$ is a stable transfer function with relative degree n_r, and $r(t)$ is a reference input. We will suppose that $|r(t)| \leq k_{r1}, \forall t \geq 0$ and $|y_m(t)| \leq k_{ym}, \forall t \geq 0$.

3 Parametrization of System and Reference Models

We now reparametrize the system and reference models so that they are in a form more suitable for the development of an adaptive control law. To do this, we need to filter the input and output signals.

Let a be a positive number satisfying $a > d_2 + 2d_0$. We define the "regression" vector[1],

$$\psi^T \quad := \quad \left(\frac{y}{(s+a)^{2n-m-1}}, \ldots, \frac{y}{(s+a)^{n-m}}, \frac{u}{(s+a)^{2n-m-1}}, \ldots, \frac{u}{(s+a)^{n-m}}\right),$$

We will presently show that there exists a "parameter vector" $\theta = (\theta_1, \ldots, \theta_{2n})^T$ such that the system (1) can be represented as,

$$y(t) = \psi^T(t)\theta + v_f'(t) + \xi_{2d_0}(t), \;\; \theta_{2n} = b_m \tag{4}$$

where $\psi^T(t)\theta$ represents the nominal part of the system, $\xi_{2d_0}(t)$ represents the effect of initial conditions arising from the filtering operations[2], and $v_f'(t)$ represents the effect of unmodeled dynamics and bounded disturbances.

[1]All of the results of this paper continue to hold if instead of the filter $\frac{1}{(s+a)^{2n-m-1}}$, we use the filter $\frac{1}{(s+a)^{n-m}\lambda_1(s)}$, where $\lambda_1(s) = (s+a)\lambda_1'(s)$ with $\lambda_1'(s)$ being monic, of degree $(n-2)$, and with all roots having real parts less than or equal to $-(2d_0 + d_2)$.

[2]Here and throughout, $\xi_q(t)$ will denote a signal which satisfies the following properties:
(i) $|\xi_q^{(i)}(t)| \leq c_0 exp[-qt], i = 0, 1$, where $\xi_q^{(i)}(t)$ denotes the i−th derivative of $\xi_q(t)$.
(ii) $\xi_q(t) \equiv 0$ when initial conditions are zero.
When the value of q is unimportant, we will sometimes drop the subscript on ξ.

We will also reparametrize the reference model (3) as

$$y_m(t) = \frac{1}{(s+a)^{n_r}} r'(t) + \xi_{2d_0}(t) \tag{5}$$

where $r'(t) := (s+a)^{n_r} W_m(s) r(t)$. Note that $r'(t)$ is well defined since the relative degree of $W_m(s)$ is n_r. Further, $r'(t)$ is bounded since $r(t)$ is bounded. We shall directly suppose from now onwards that $r'(t) \leq k_r, \forall t \geq 0$.

To see the existence of a θ for which (4) holds, let $F(s)$ be a monic polynomial of degree $(n_r - 1)$ and $G(s)$ a polynomial of degree less than or equal to $(n-1)$, such that $A(s)F(s) + G(s) = \lambda(s)$, where $\lambda(s) := (s+a)^{2n-m-1}$. Then, from (1), taking into consideration the effect of the initial conditions introduced by the filtering operation $\frac{1}{\lambda(s)}$, we have,

$$\begin{aligned} \frac{\lambda(s) - G(s)}{\lambda(s)} y(t) &= F(s) \frac{A(s)}{\lambda(s)} y(t) + \xi_{2d_0}(t) \\ &= F(s)[\frac{B(s)}{\lambda(s)} u(t) + \mu_m \frac{B(s)\Delta_m(s)}{\lambda(s)} u(t) + \frac{A(s)}{\lambda(s)} v(t)] + \xi_{2d_0}(t). \end{aligned}$$

Thus, if $\theta = (\theta_1, \ldots, \theta_{2n})^T$ is defined from the coefficients of $G(s)$ and $B(s)F(s)$ by, $\theta_n(s+a)^{n-1} + \ldots + \theta_2(s+a) + \theta_1 = G(s)$, $\theta_{2n}(s+a)^{n-1} + \ldots + \theta_{n+2}(s+a) + \theta_{n+1} = B(s)F(s)$, and $v'_f(t) = \frac{A(s)F(s)}{\lambda(s)} v(t) + \mu_m \frac{B(s)F(s)}{\lambda(s)} \Delta_m(s) u(t)$, then (4) is satisfied. We note that since F is monic, $\theta_{2n} = b_m$.

4 The Adaptive Control Law

We will use the control law,

$$\phi^T(t)\hat{\theta}(t) = r'(t), \tag{6}$$

to implicitly define the input $u(t)$, where $\phi(t) := (\frac{y}{(s+a)^{n-1}}, \ldots, y, \frac{u}{(s+a)^{n-1}}, \ldots, u)^T$, and $\hat{\theta}(t)$ is an estimate of θ that we shall presently specify. Note that the "regression" vector $\psi(t)$ defined earlier satisfies $\psi(t) = \frac{1}{(s+a)^{n-m}} \phi(t)$.

The adaptive control law (6) is a "certainty equivalent" control law, since if $\hat{\theta}(t) = \theta$ in (6), then the nominal part of the system (4) tracks $y_m(t)$ since,

$$\begin{aligned} y(t) &= \psi^T(t)\theta + \xi_{2d_0}(t) = \frac{1}{(s+a)^{n-m}} [\phi^T(t)\theta] + \xi_{2d_0}(t) \\ &= \frac{r'(t)}{(s+a)^{n-m}} + \xi_{2d_0}(t) = y_m(t) + \xi_{2d_0}(t). \end{aligned}$$

For parameter estimation, we shall use the gradient update law with projection,

$$\dot{\hat{\theta}}(t) = \text{Proj}[\hat{\theta}(t), \frac{\alpha\psi(t)e_a(t)}{n(t)}], \ \|\hat{\theta}(0)\| \leq M, \ \hat{\theta}_{2n}(0) \geq b_{min}, \tag{7}$$

where $\alpha > 0$, $e_a(t)$ is the "augmented" error,

$$\begin{aligned} e_a(t) &= y(t) - y_m(t) + \frac{1}{(s+a)^{n-m}}[\phi^T(t)\hat{\theta}(t)] - \psi^T(t)\hat{\theta}(t), \\ n(t) &:= \nu + \sum_{i=1}^{2n-m-1}([\frac{y(t)}{(s+a)^i}]^2 + [\frac{u}{(s+a)^i}]^2), \ \nu > 0, \end{aligned}$$

is a normalizer of the gradient, and $\text{Proj}[\cdot,\cdot]$ is a projection whose $i-$th component is defined for $i \neq 2n$, by

$$\text{Proj}[p,x]\mid_i = \begin{cases} x_i & \text{if } \|p\| < M \text{ or } p^T x \leq 0, \\ x_i - \frac{p^T x}{\|p\|^2} p_i & \text{otherwise,} \end{cases}$$

and for $i = 2n$ by

$$\text{Proj}[p,x]\mid_{2n} = \begin{cases} x_{2n} & \text{if } (\|p\| < M \text{ or } p^T x \leq 0) \\ & \text{and } (p_{2n} > b_{min} \text{ or } x_{2n} \geq 0), \\ x_{2n} - \dfrac{p^T x}{\|p\|^2} p_{2n}\,, & \text{if } (\|p\| \geq M \text{ and } p^T x > 0) \\ & \text{and } (p_{2n} > b_{min} \text{ or } x_{2n} \geq 0), \\ 0\,, & \text{otherwise.} \end{cases}$$

There are several features worth commenting upon. First, the augmented error $e_a(t)$ consists of the tracking error $y(t) - y_m(t)$, as well as the familiar "swapping" term $\frac{1}{(s+a)^{n-m}}[\phi^T(t)\hat{\theta}(t)] - [\frac{1}{(s+a)^{n-m}}\phi^T(t)]\hat{\theta}(t)$, which would be zero if $\hat{\theta}(t)$ were a constant. Second, note that $n(t)$ is slightly different from the usual $\|\psi(t)\|^2$ since it additionally contains the lower order filtered terms $\frac{y(t)}{(s+a)^i}$ and $\frac{u(t)}{(s+a)^i}$ for $1 \leq i \leq n-m-1$, which are absent in $\psi(t)$.

Turning to the "projection" mechanism, it has two features. Without any projection, a gradient scheme would simply consist of,

$$\dot{\hat{\theta}}(t) = \frac{\alpha\psi(t)e_a(t)}{n(t)}.$$

However, to keep the estimate inside a sphere of radius M centered at the origin, when the estimate is about to leave the sphere, one would project the drift term so that it evolves tangentially to the sphere. Our projection $\text{Proj}[p,x]$ also involves an additional feature to ensure that $\hat{b}_m(t) = 2n-$th component of $\hat{\theta}(t)$ is larger than or equal to b_{min}.

The following easily verifed consequences of projection are important.

(P0) $\hat{\theta}_{2n}(t) \geq b_{min}, \|\hat{\theta}(t)\| \leq M$.

(P1) $\|\dot{\hat{\theta}}(t)\| \leq |\frac{\alpha\psi(t)e_a(t)}{n(t)}|$

(P2) If $\|\theta\| \leq M$ and $\theta_{2n} = b_m \geq b_{min}$, i.e. θ lies in the region to which we confine the parameter estimates, then

$$\tilde{\theta}^T(t)\dot{\hat{\theta}}(t) \leq \tilde{\theta}^T(t)(\frac{\alpha\psi(t)e_a(t)}{n(t)}) \qquad \text{where } \tilde{\theta}(t) := \hat{\theta}(t) - \theta.$$

Finally, since the existence of a solution to the above parameter estimator may not be assured due to the discontinuous nature of the projection, we can replace the above projection by the "smooth" projection due to Pomet and Praly [15], for which existence is assured. The key properties (P1) and (P2) continue to hold for the "smooth" projection, and all the results of this paper rigorously hold for this "smooth" projection.

In what follows, we will throughout suppose that the nominal plant described by θ satisfies the assumptions of (P2).

5 Bounding Signals by m

In this section we will show that all the signals in the system can be bounded in terms of $m(t)$ given in (2).[3]

For brevity of notation in the proof, we define $\lambda_1(s) = (s+a)^{n-1}, \lambda_2(s) = (s+a)^{n_r}, \lambda(s) = \lambda_1(s)\lambda_2(s), \zeta(t) = \frac{1}{\lambda_2(s)}[\phi^T(t)\hat{\theta}(t)] - [\frac{1}{\lambda_2(s)}\phi^T(t)]\hat{\theta}(t)$, $\psi'(t) = [\frac{y(t)}{\lambda(s)}, \ldots, \frac{y(t)}{(s+a)}, \frac{u(t)}{\lambda(s)}, \ldots, \frac{u(t)}{(s+a)}]^T$, $v_f(t) = \frac{B(s)F(s)}{\lambda(s)}v(t)$, and $e_1(t) = y(t) - y_m(t)$.

Lemma 5.1.

$$|u(t)| \leq K_{u\psi'}\|\psi'(t)\| + K_{uv}|v_f'(t)| + k_{uu} + c_0 exp[-2d_0 t], \forall t \geq 0. \tag{8}$$

[3]Here and throughout, the values of useful constants used in the bounds are specified in Table C in the Appendix. Certain constants whose exact value is unimportant and which do not depend on K_v, μ_m, k_v will be denoted generically by C. Any constant whose exact value is unimportant but which depends on K_v, μ_m, k_v, and such that its value decreases as K_v, μ_m, k_v decrease, will be denoted generically by c. Any positive constant which depends only on initial conditions of some filter, and whose exact value is unimportant will be denoted by c_0. Finally, all constants throughout are positive, unless otherwise noted.

Proof. Let $\phi_I := (\frac{y}{\lambda(s)}, \ldots, \frac{y}{(s+a)}, y, \frac{u}{\lambda(s)}, \ldots, \frac{u}{(s+a)})$ be a sub-vector of ϕ, and $\hat{\theta}_I$ the corresponding sub-vector of $\hat{\theta}$. Then, the control law (6) gives $u = (r' - \phi_I^T \hat{\theta}_I^T)/\hat{b}_m$. From (4), we then obtain $|y(t)| \leq M\|\psi'\| + |v'_f| + c_0 exp[-2d_0 t]$. Hence $\|\phi_I\| \leq |y| + \|\psi'\| \leq (1+M)\|\psi'\| + |v'_f| + c_0 exp[-2d_0 t]$. Since $\hat{b}_m \geq b_{min}$, and $\|\hat{\theta}_I\| \leq \|\hat{\theta}\| \leq M$, we thus conclude that $|u| \leq \frac{1}{b_{min}}(k_r + M(1+M)\|\psi'\| + M|v'_f|) + c_0 M exp[-2d_0 t]$. □

Theorem 5.1.

(i) $|v_f(t)| \leq K_{vf} m(t) + k_{vf} + c_0 exp[-(d_0 + d_2/2)t] + k_{vf0} exp[-pt], \forall t \geq 0,$

(ii) $|v'_f(t)| \leq K'_{vf} m(t) + k_{vf} + c_0 exp[-(d_0 + d_2/2)t] + k_{vf0} exp[-pt], \forall t \geq 0,$

(iii) $\|\psi'(t)\| \leq K_{\psi' m} m(t) + c_0 exp[-2d_0 t], \forall t \geq 0,$

(iv) $|e_a(t)| \leq K_{em} m(t) + k_{vf} + c_0 exp[-(d_0 + d_2/2)t] + k_{vf0} exp[-pt] + c_0 exp[-2d_0 t], \forall t \geq 0,.$

(v) $|y(t)| \leq K_{ym} m(t) + k_{vf} + c_0 exp[-(d_0 + d_2/2)t] + k_{vf0} exp[-pt] + c_0 exp[-2d_0 t], \forall t \geq 0,$

(vi) $|u(t)| \leq K_{um} m(t) + k_{um} + c_0 exp[-(d_0 + d_2/2)t] + K_{uv} k_{vf0} exp[-pt] + c_0 exp[-2d_0 t], \forall t \geq 0,$

The proof of this Theorem is based on the following Lemma.

Lemma 5.2. *Let $H(s)$ be a strictly proper, stable transfer function, whose poles $\{p_j\}$ satisfy $Re(p_j) \leq -(d_0 + d_2) < 0$ for all j.*
(i) If $w_{in}(t)$ is the input to a system with transfer function $H(s)$, which satisfies the bound $|w_{in}(t)| \leq k_u|u(t)| + k_y|y(t)| + k_m m(t) + k' + k'' exp[-p't], \forall t \geq 0$, for $0 < p' \leq d_0$, then the output $w_{out}(t)$ is bounded by,
$|w_{out}(t)| \leq k_1\|x(0)\| exp[-(d_0 + d_2/2)t] + k_3 m(t) + k_4 + k_5 exp[-p't]$, where $x(0)$ is the initial state corresponding to a minimal state representation of $H(s)$, and k_1, k_3, k_4 and k_5 are related positive constants specified below.
(ii) Let $H'(s) = k_p + H(s)$. If $w_{in}(t)$ is the input to a system with transfer function $H'(s)$, which satisfies the bound $|w_{in}(t)| \leq k_m m(t) + k' + k'' exp[-p't]$, $\forall t \geq 0$, for $0 < p' \leq d_0$, then the output $w_{out}(t)$ is bounded by $|w_0(t)| \leq k_1\|x(0)\| exp[-(d_0 + d_2/2)t] + k_6 m(t) + k_7 +$

$k_8 exp[-p't]$, *where* $x(0)$ *is the initial state corresponding to a minimal state representation of* $H'(s)$, *and* k_1, k_6, k_7 *and* k_8 *are related positive constants specified below.*[4]

Proof. See [20]. □

Proof of Theorem 5.1. The results (i) and (ii) follow from Lemma 5.2 and (A4) by noting that $v_f = \frac{A(s)F(s)}{\lambda(s)}v$ and $v_f' = v_f + \mu_m \frac{B(s)F(s)}{\lambda(s)}\Delta_m(s)u$. The result (iii) follows by applying Lemma 5.2 to each component of ψ', noting that each component is either of the form $\frac{y}{(s+a)^k}$ or $\frac{u}{(s+a)^k}$. Since $e_a = -\psi^T\tilde{\theta} + v_f' + \xi_{2d_0}$ the bound for e_a in (iv) follows from the bounds for $\|\psi\| \leq \|\psi'\|$ and v_f'. The proof of (v) is similar to (iv) since $y = \psi^T\theta + v_f' + \xi_{2d_0}$. Finally, (vi) follows from Lemma 5.1 and (ii), (iii) above. □

6 Bounding by a Switched System

Let us introduce the "switched" system,

$$\dot{z}(t) = I(t)[-d_0 z(t) + d_1(|u(t)| + |e_a(t)| + 1)] + (1 - I(t))[-g_2 z(t) + K_2], \quad z(0) \geq \frac{m(0)}{(1 + \frac{2\alpha K_{u\psi'}}{(a-2d_0)})} \tag{9}$$

$$\begin{aligned} \text{where } I(t) &= 1 \text{ if } (-d_0 z(t) + d_1(|u(t)| + |e_a(t)| + 1)) \geq (-g_2 z(t) + K_2), \\ &= 0 \text{ otherwise.} \end{aligned} \tag{10}$$

In Theorem 6.1(ii) below we will show that m itself is bounded in terms of z, thus showing from Theorem 5.1 that all other signals can also be bounded in terms of z.

Theorem 6.1.

$$(i) \quad \dot{m} \leq K_m m + k_m + c_0 exp[-(d_0 + d_2/2)t] + c_0 exp[-pt] + c_0 exp[-2d_0 t], \tag{11}$$

$$(ii) \quad m(t) \leq K_{mz} z(t) + k_{mz}, \quad \forall t \geq 0, \tag{12}$$

$$(iii) \quad \dot{z} \leq K_z z + k_z + c_0 exp[-(d_0 + d_2/2)t] + c_0 exp[-pt] + c_0 exp[-2d_0 t]. \tag{13}$$

[4] k_3, k_4, k_6 and k_7 are independent of k'', which will mean that initial conditions have no influence on the magnitude of the bounds or the size of allowable unmodeled dynamics.

Proof. (i) This follows from $\dot{m} = -d_0 m + d_1(|u| + |y| + 1)$, by making use of the bounds for $|u|$ and $|y|$ in terms of m given in Theorem 5.1(v),(vi).

(ii) By the definition of the augmented error e_a, we have $y = (e_a + y_m) - \zeta$, which implies $|y| \leq |e_a| + |y_m| + |\zeta|$. By the Swapping Lemma (Morse [6]; Goodwin & Mayne [13]), $\zeta(t) = [h^T exp(Q_1 t)] * (H(t)\dot{\tilde{\theta}}(t))$, where Q_1 is an $(n_r \times n_r)$ stable matrix such that $det(sI - Q_1) = \lambda_2(s)$, $H^T(t) = (\psi(t), s\psi(t), \ldots, s^{n-m-1}\psi(t))$, h^T is a constant row vector of dimension n_r, and "$*$" denotes convolution. Now,

$$\|H^T \dot{\tilde{\theta}}\| = \|H^T Proj(\hat{\theta}, \frac{\alpha \psi e_a}{\nu + \|\psi'\|^2})\| \leq \alpha C |e_a|, \tag{14}$$

where the last inequality follows from Property (P1) of the parameter estimator, and since $\|H\| \leq C\|\psi'\|$. Since $\lambda_2(s) = (s+a)^{n_r}$, we have $\|h^T exp(Q_1 t)\| \leq C exp(-at/2)$. Therefore $|\zeta(t)| \leq \int_0^t \alpha C exp[-a(t-\tau)/2]|e_a(\tau)|d\tau$, and so,

$$\begin{aligned}
\int_0^t |\zeta| exp[-d_0(t-\tau)]d\tau &\leq C\alpha \int_0^t exp[-d_0(t-\tau)] \int_0^\tau exp[-a(\tau - t')/2]|e_a(t')|dt' d\tau \\
&= C\alpha \int_0^t exp[-d_0 t] exp[at'/2] \int_{t'}^t exp[-(a/2 - d_0)\tau]|e_a(t')|d\tau dt' \\
&\leq \frac{2C\alpha}{a - 2d_0} \int_0^t exp[-d_0(t-t')]|e_a(t')|dt'.
\end{aligned}$$

This implies

$$\int_0^t |y| exp[-d_0(t-\tau)]d\tau \leq \int_0^t exp[-d_0(t-t')](1 + \frac{2C\alpha}{(a-2d_0)})|e_a(t')|dt' + \frac{k_{ym}}{d_0}$$

Now, add $\int_0^t (|u(\tau)| + 1) exp[-d_0(t-\tau)]d\tau$ to both sides and multiply by d_1. Finally adding $exp(-d_0 t)m(0)$ to both sides yields the desired result.

(iii) The result follows from $\dot{z}(t) \leq -(d_0 + g_2)z(t) + d_1(|u| + |e_a| + 1) + K_2$ and the bounds on $|e_a|$ and $|u|$ in Theorem 5.1(iv, vi) by using the bound for m in (ii). □

From Theorem 6.1 it follows that z can grow at most exponentially fast, and therefore does not have a finite escape time. Since z bounds all signals, it follows that no signal in the system has a finite escape time.

In what follows, we choose a large time T_l, and restrict attention to $t \geq T_l$, for which the following bounds hold.

$$\text{(i)} \quad |v(t)| \leq K_v m(t) + k_v + k(T_l) \leq K_v K_{mz} z(t) + K_v k_{mz} + k_v + k(T_l). \tag{15}$$

(ii) $|v_f'(t)| \le K_{vf}' K_{mz} z(t) + K_{vf}' k_{mz} + k_{vf} + k(T_l).$ (16)

(iii) $\|\psi'(t)\| \le K_{\psi' m} K_{mz} z(t) + K_{\psi' m} k_{mz}.$ (17)

(iv) $|y(t)| \le M\|\psi'(t)\| + K_{vf}' K_{mz} z(t) + K_{vf}' k_{mz} + k_{vf} + k(T_l).$ (18)

(iv)′ $|y(t)| \le K_{yz} z(t) + k_{yz} + k(T_l).$ (19)

(v) $|u(t)| \le K_{u\psi'}\|\psi'(t)\| + K_{vf}' K_{uv} K_{mz} z(t) + k_u + k(T_l).$ (20)

(v)′ $|u(t)| \le K_{uz} z(t) + k_{uz} + k(T_l).$ (21)

(vi) $\dot{z}(t) \le K_z z(t) + k_z + k(T_l).$ (22)

(vii) $\eta_m(t) := \dfrac{\Delta_m(s)}{q(s)} u(t)$ satisfies $|\eta_m(t)| \le cz(t) + c + k(T_l),$ (23)

if $q(s - d_0 - d_2)$ is a Hurwitz polynomial. (The result (vii) follows from Theorem 5.1(vi), Lemma 5.2(ii) and Theorem 6.1(ii).) In all the above inequalities and in what follows, $k(T_l)$ denotes a generic constant which decreases with increasing T_l at a rate faster than $exp[-pT_l]$.

7 Comparing ψ' and z

In this section we compare ψ' and z.

Theorem 7.1. *(i) If $I(t) = 1$, then $\frac{n(t)}{z(t)^2} \ge K_{nz}$.*
(ii) Consider $T' > 0$. Let $t_1 \ge T_l + T'$ be any instant such that $I(t_1) = 1$. If $z(t) \ge \frac{L}{2}, \forall t \in (t_1 - T', t_1]$ where L is a large positive constant (specified in Table C), then there exists a $K_{vmax} > 0$ and a $\mu_{max} > 0$ such that for all $K_v \in [0, K_{vmax}]$ and for all $\mu_m \in [0, \mu_{max}]$, we have:

$$\frac{n(t)}{z^2(t)} > \delta(T'), \quad \forall t \in (t_1 - T', t_1] \tag{24}$$

Proof. (i) Let us fix t such that $I(t) = 1$. By (10), we have $|u| + |e_a| \ge \frac{d_0 - g_2}{d_1} z + \frac{K_2 - d_1}{d_1}$. Using the upper bound (20) on u gives, $K_{u\psi'}\|\psi'\| + |e_a| \ge (\bar{d}_0 - \bar{g}_2)z + \bar{K}_2 =: RHS$. We note that we can take $\bar{d}_0 \ge 0$ and $\bar{K}_2 \ge 0$, by the choices of K_2, T_l, K_{vmax}, and μ_{max} given in Table C. If $\|\psi'\| > \frac{RHS}{2K_{u\psi'}}$, then the result clearly holds, so we consider $\|\psi'\| \le \frac{RHS}{2K_{u\psi'}}$. Then clearly $|e_a| \ge \frac{RHS}{2}$. Therefore using the upper bound (16) for v_f', we obtain by definition of the augmented error,

$$\begin{aligned}\tilde{M}\|\psi\| &\geq |\tilde{\theta}^T\psi| \geq |e_a| - |v_f'| - c_0 exp[-2d_0 T_l], \\ &\geq |[(\bar{d}_0 - \bar{g}_2)z + \bar{K}_2]/2| - (K_{vf}' K_{mz} z(t) + K_{vf}' k_{mz} + k_{vf} + k(T_l) + c_0 exp[-pT_l])\end{aligned}$$

Note that $(\bar{d}_0 - \bar{g}_2) \geq 4K_{vf}' K_{mz}$ and $\bar{K}_2 \geq 2(K_{vf}' k_{mz} + k_{vf} + k(T_l) + k exp[-pT_l])$, again from the choices of K_2, T_l, K_{vmax}, and μ_{max} given in Table C. This implies $\|\psi'\| \geq \|\psi\| \geq [\frac{(\bar{d}_0 - \bar{g}_2)z}{4\tilde{M}}]$. Hence we obtain, $\frac{n(t)}{z(t)^2} \geq K_{nz}$.

(ii) First, we will bound the growth rate of $\frac{n(t)}{z^2(t)}$. We do not require z to be large for this part of the proof. Note that $\frac{d}{dt}(\frac{n(t)}{z^2(t)}) = \frac{\dot{n}}{z^2} - 2\frac{n}{z^2}\frac{\dot{z}}{z} \leq \frac{\dot{n}}{z^2} + 2g_2\frac{n}{z^2}$. Also, $\dot{\psi}' = \phi' - a\psi'$, which implies $\dot{n} = 2\psi'^T(\phi' - a\psi') \leq (1-2a)\|\psi'\|^2 + \|\phi'\|^2$, where, $\phi'^T := (s+a)\psi'^T = (\frac{y}{(s+a)^{2n-m-2}}, \ldots, \frac{y}{(s+a)}, y, \frac{u}{(s+a)^{2n-m-2}}, \ldots, \frac{u}{(s+a)}, u)$. Now, $\|\phi'\|^2 \leq \|\psi'\|^2 + y^2 + u^2$. Using this, we get $\dot{n} \leq 2(1-a)\|\psi'\|^2 + y^2 + u^2$. Next, using the upper bounds for u and y from (20), (18), and recalling that $n(t) = \nu + \|\psi'(t)\|^2$, we get $\frac{d}{dt}(\frac{n(t)}{z^2(t)}) \leq K_a(\frac{n(t)}{z^2(t)}) + K_b + \frac{K_c}{z^2(t)}$.

Next, using the fact that $z(t) \geq \frac{L}{2}, \forall t \in (t_1 - T', t_1]$, we get $\frac{d}{dt}(\frac{n(t)}{z^2(t)}) \leq K_a(\frac{n(t)}{z^2(t)}) + K_d$ where $K_d := K_b + \frac{4K_c}{L^2}$. This gives,

$$\begin{aligned}\frac{n}{z^2}(t_1) &\leq exp[K_a(t_1 - t')][\frac{n(t')}{z^2(t')} + \int_{t'}^{t_1} exp[K_a(t' - \tau)]K_d d\tau] \\ &\leq exp[K_a T'][\frac{n(t')}{z^2(t')} + \frac{K_d}{K_a}(1 - exp[-K_a T'])],\ \forall t' \in [t_1 - T', t_1].\end{aligned}$$

Thus, $\frac{n(t')}{z^2(t')} \geq exp[-K_a T'][\frac{n}{z^2}(t_1) + \frac{K_d}{K_a}] - \frac{K_d}{K_a} \geq exp[-K_a T'][K_{nz} + \frac{K_d}{K_a}] - \frac{K_d}{K_a}$, $\forall t' \in [t_1 - T', t_1]$. The fact that there exist $K_{vmax} > 0, \mu_{max} > 0$ such that $\delta(T') > 0$ for all $K_v \in [0, K_{vmax}], \mu_m \in [0, \mu_{max}]$ follows from the fact that $\frac{K_d}{K_a}$ can be made as small as we wish by making L sufficiently large, and then choosing appropriately small K_{vmax} and μ_{max}. □

8 A Nonminimal System Representation

Recalling $y = \theta^T\psi + v_f' + \xi_{2d_0}$, $\psi(t) = \frac{1}{\lambda_2(s)}\phi(t)$, $\phi^T(t)\hat{\theta}(t) = r'(t)$, and $v_m = \frac{BF}{\lambda}\Delta_m u$ we note that,

$$y = \frac{1}{\lambda_2(s)}(r' - \tilde{\theta}^T\phi) + v_f + \mu_m v_m + \xi_{2d_0} \tag{25}$$

The system (1) can be represented as,

$$\dot{x}_p = A_p x_p + b_p u + \mu_m b_p' \eta_m \quad , \quad y = h_p^T x_p + v + \xi_{2d_0} \tag{26}$$

where (A_p, b_p, h_p^T) is a minimal state representation of the *nominal* plant transfer function, $\eta_m := \frac{\Delta_m(s)}{q(s)} u$, with $q(s)$ chosen as some polynomial of degree $n_r - 1$ such that $q(s - 2d_0 - d_2)$ is Hurwitz, thus giving rise to the ξ_{2d_0} term in (26)). In (26) above, $h_p^T(sI - A_p)^{-1} b_p = \frac{B(s)}{A(s)}$ and $h_p^T(sI - A_p)^{-1} b_p' = \frac{B(s)q(s)}{A(s)}$. Note that $\frac{\Delta_m(s)}{q(s)}$ is a proper, stable transfer function. The control law (6) is equivalent to $\theta^T \phi + \tilde{\theta}^T \phi = r'$. Hence,

$$b_m u \;\; = \;\; -\theta_u^T \phi_u - \theta_n y - \theta_y^T \phi_y - \tilde{\theta}^T \phi + r'. \tag{27}$$

Defining $\phi_y = (\frac{y}{(s+a)^{n-1}}, \ldots, \frac{y}{(s+a)})^T$ and ϕ_u like-wise, we note that $\phi(t) = (\phi_y^T(t), y(t), \phi_u^T(t), u(t))^T$, with

$$\dot{\phi}_y(t) = Q\phi_y(t) + qy(t), \text{ and } \dot{\phi}_u(t) = Q\phi_u(t) + qu(t), \tag{28}$$

where Q is stable matrix such that $det(sI - Q) = (s + a)^{n-1}$, and (Q, q) is a controllable pair.

Define $X_c^T := [x_p^T, \phi_y^T, \phi_u^T]$. Then, using (26), (28), and (27), we get $\dot{X}_c = A_c X_c + b_c(r' - \tilde{\theta}^T \phi) + b_{cv} v + \mu_m b_{c\eta} \eta_m, y = h_c^T X_c + v + \xi_{2d_0}$, where, A_c is a stable matrix (as we will prove shortly), $b_c^T := [b_p^T / b_m, 0, q^T / b_m], b_{cv}^T := [-\theta_n b_p^T / b_m, q^T, -\theta_n q^T / b_m], b_{c\eta}^T := [b_p'^T, 0, 0]$ and $h_c^T := [h_p^T, 0, 0]$; see [3].

For $\tilde{\theta} = 0, v \equiv 0$ and $\mu_m = 0$, we have, $\dot{X}_c = A_c X_c + b_c r', y = h_c^T X_c + \xi_{2d_0}$. This implies $y = h_c^T(sI - A_c)^{-1} b_c r' + \xi_{2d_0}$. However, we already know (from (25)) that $y = \frac{1}{\lambda_2(s)} r' + \xi_{2d_0}$, in such a case. This means that, (after cancellations),

$$h_c^T(sI - A_c)^{-1} b_c = \frac{1}{\lambda_2(s)} \tag{29}$$

It can be easily verified (e.g. see [17], pg.136) that we have only stable pole-zero cancellations because the nominal plant is minimum-phase and Q is a stable matrix. This proves that (A_c, h_c^T) is detectable. Since the overall transfer function is stable (by (29)), we conclude that A_c is a stable matrix.

Thus we can write the following non-minimal representation of $\frac{1}{\lambda_2(s)}$:

$$\dot{X}_m = A_c X_m + b_c r' \quad , \quad y_m = h_c^T X_m$$

where $X_m^T := [x_m^T, \phi_{ym}^T, \phi_{um}^T]$. A_c being stable and r' being bounded implies that $\|X_m(t)\| \leq K_{Xm}, \forall t \geq 0$, for some positive constant K_{Xm}. Define the *state error* e as $e := X_c - X_m$. This gives,

$$\dot{e} = A_c e - b_c(\tilde{\theta}^T \phi) + b_{cv} v + \mu_m b_{c\eta} \eta_m \quad , \quad e_1 = h_c^T e + v \tag{30}$$

where we recall that $e_1 = y - y_m$ is the tracking error.

9 Robust Ultimate Boundedness

Define W, which bounds all signals, by

$$W = k_e e^T P e + \frac{1}{2} z^2 \tag{31}$$

where $P = P^T > 0$ satisfies $PA_c + A_c^T P = -I$. Such a P exists since A_c is a stable matrix.

Our main result on robust ultimate boundedness of the overall system is given by the following Theorem. It states that eventually $W(t)$ (which bounds all signals) enters a compact set, the size of which is independent of the initial conditions. Further, the size of the allowable unmodeled dynamics for which this is guaranteed, is independent of the initial conditions. It should be noted, however, that the time T_{large} that it takes for $W(t)$ to enter this compact set can depend on the initial conditions.

Choose constants $0 < \gamma < 1$, $\epsilon_0 > 0$ small, and $\epsilon_z > 0$. Let $T, \epsilon, T_l, K_{vmax}$, and μ_{max} be as in Table C.

Theorem 9.1 (Robust Ultimate Boundedness Theorem).

There exist a $T_{large} \geq T_l$, and positive K_{vmax}, μ_{max}, such that for all $K_v \in [0, K_{vmax}]$, and for all $\mu_m \in [0, \mu_{max}]$,

$$W(t) \leq 8K_{wz}^2 L^2 exp[4(K_z + \epsilon_z)T], \quad \forall t \geq T_{large} \tag{32}$$

Furthermore, K_{vmax}, μ_{max}, T, and L are independent of the initial conditions.

Proof. The idea of the proof, based on the following Lemmas, is to show that whenever $W(t) \geq K_{wz}L^2$ throughout an interval of length $2T$, then at the end of the interval its value is smaller than at the beginning of the interval.

Lemma 9.1.

$$W(t) \leq K_{wz}z^2(t) + k_{Wz}, \forall t \geq T_l. \tag{33}$$

Proof. Since $W = e^T Pe + \frac{1}{2}z^2, e = X_c - X_m, \|X_m\| \leq K_{Xm}, X_c = [x_p^T, \phi_y^T, \phi_u^T]^T, \|\phi_y\| \leq cz, \|\phi_u\| \leq cz$, it suffices to show that $\|x_p\| \leq cz + c + c_0 exp[-pt]$.

By (26), defining $u' := \frac{1+\mu_m \Delta_m(s)}{q(s)}u$, $y' = y - v - \xi_{2d_0}(t)$, and b_p'' such that $h_p^T(sI - A_p)^{-1}b_p'' = \frac{B(s)q(s)}{A(s)}$, with the (ξ_{2d_0}) term in the above equation possibly different from the (ξ_{2d_0}) term in (26), we get, $\dot{x}_p = A_p x_p + b_p'' u'$, $y' = h_p^T x_p$. Now, from Theorem 5.1(v), $|y'(t)| \leq cm(t) + c + c_0 exp[-pt], \forall t \geq 0$. Noting that $B'(s) := B(s)q(s)$ is such that $B'(s-p)$ is Hurwitz, and applying Lemma A.1, we get $\|x_p(t)\| \leq cm(t) + c + c_0 exp[-pt], \forall t \geq 0$. Finally applying Theorem 5.2(ii), we get the desired bound on $\|x_p\|$. □

Lemma 9.2. *Consider an interval* $[a, b]$. *Then*

$$\frac{W(b)}{W(a)} \leq exp[\int_a^b g(\tau)d\tau](1 + \frac{\beta_4 exp[\beta(b-a)]}{\beta W(a)}) \tag{34}$$

where $g(t) := -\beta + \beta_3 \frac{|\tilde{\theta}^T(t)\phi(t)|}{z(t)} + \beta_5 \frac{|\tilde{\theta}^T(t)\psi(t)|}{z(t)}$, *and* β_3, β_4 *and* β_5 *are specified in [20].*

Proof. Recalling from (9) that $\dot{z} \leq -(d_0 + g_2)z + d_1(|u| + |e_a| + 1) + K_2$, we have,

$$\begin{aligned}
\dot{W} &= k_e(\dot{e}^T Pe + e^T P\dot{e}) + \dot{z}z \\
&= -k_e\|e\|^2 - (d_0 + g_2)z^2 + \\
&\quad 2k_e(-(\tilde{\theta}^T\phi)b_c^T Pe + vb_{cv}^T Pe + \mu_m \eta_m b_{c\eta}^T Pe) + d_1(|u| + |e_a| + 1)z + K_2 z
\end{aligned}$$

using (30). Then, proceeding as in Ioannou & Tsakalis [12]((4.16)-(4.18)), (with the exception being that now, we also need to take care of a linear term in $\|e\|$) we get:

$$\begin{aligned}
\dot{W}(t) &\leq -\beta W(t) + \beta_3 \frac{|\tilde{\theta}^T(t)\phi(t)|}{z(t)} W(t) + \beta_5 \frac{|\tilde{\theta}^T(t)\psi(t)|}{z(t)} W(t) + \beta_4 \\
&=: g(t)W(t) + \beta_4, \text{ which implies} \\
\frac{W(b)}{W(a)} &\leq exp[\int_a^b g(\tau)d\tau](1 + \frac{\beta_4 exp[\beta(b-a)]}{\beta W(a)}) \text{ since } -g(\tau) \leq \beta.
\end{aligned}$$

Details of the proof are provided in [20]. □

Lemma 9.3. *Consider a time interval $[a,b]$ such that (i) $b \geq a \geq T_l$, (ii) $W(t) \geq K_{wz}L^2, \forall t \in [a,b]$. Then,*

(i) $\frac{1}{2}z^2(t) \leq W(t) \leq 4K_{wz}z^2(t), \forall t \in [a,b]$.

(ii) If $I(t) = 0, \forall t \in [a,b]$, *then,* $W(b) \leq 8K_{wz}(exp[-g_2(b-a)] + \frac{2K_2}{g_2L})^2W(a)$.

(iii) If $(b-a) \geq 1$ *and for each* $t \in [a,b]$ *such that* $I(t) = 0$ *there exists a* $t' \in [0,T]$ *such that* $I(t+t') = 1$, $W(b) \leq k \cdot exp[-\frac{\beta}{4}(b-a)](1 + \frac{\beta_4}{K_{wz}\beta L^2}exp[\beta(b-a)])W(a)$.

(iv) $W(b) \leq 8K_{wz}exp[2(K_z + \epsilon_z)(b-a)]W(a)$.

Proof. The result (i) follows by Lemma 9.1 since $W \geq K_{wz}L^2$ implies $z \geq L/2$. The result (ii) follows from (i) because of $\dot{z} \leq -g_2z + K_2$, and since $W \geq K_{wz}L^2$ implies $z \geq L/2$. To prove (iii), note from Lemmas 9.2, B.3, B.4 and by our choices of $\epsilon, \mu_{max}, K_{vmax}$ and L, that

$$\begin{aligned}\int_a^b g(t)dt &\leq -\beta(b-a) + \beta_3\nu_3/\epsilon^2 + \beta_3[\mu^2\nu_4(T)/\epsilon^2 + \mu\nu_5(T)/\sqrt{\epsilon} + \nu_6(T)\epsilon^\rho \\ &\quad +4a^{n-m}k(\mu,L)](b-a) + \beta_5\nu_7/\epsilon^2 + \beta_5[\mu^2\nu_8(T)/\epsilon^2 + \sqrt{\epsilon}](b-a) \\ &\leq -\frac{\beta}{4}(b-a) + \frac{(\beta_3\nu_3 + \beta_5\nu_7)}{\epsilon^2}\end{aligned}$$

which gives us the desired result after defining $k = exp[\frac{(\beta_3\nu_3+\beta_5\nu_7)}{\epsilon^2}]$. Finally, (iv) follows from the bounded growth-rate of $z(t)$ shown in (22), and (i). □

Lemma 9.4 (Contraction Property).

If $t_0 \geq T_l$, and $W(t) \geq K_{wz}L^2\ \forall t \in [t_0, t_0 + 2T]$, then $W(t_0 + 2T) \leq \gamma W(t_0)$.

Proof. There are four possibilities:

Case 1 Suppose $I(t) = 0, \forall t \in [t_0, t_0 + 2T]$. Then from Lemma 9.3(ii), we have $W(t_0 + 2T) \leq 8K_{wz}(exp[-2g_2T] + \frac{2K_2}{g_2L})^2W(t_0) \leq \gamma W(t_0)$, where the last inequality follows by the definitions of T and L given in Table C.

Let $t_1 := \min\{t \in [0,2T] : I(t_0 + t) = 1\}$ and $t_2 := \max\{t \in [0,2T] : I(t_0 + t) = 1\}$.

Case 2 Suppose $0 \leq t_1 \leq t_2 < T$. First, by Lemma 9.3(ii), $W(t_0+2T) \leq 8K_{wz}(exp[-g_2(2T-t_2)] + \frac{2K_2}{g_2L})^2 W(t_0+t_2)$.

(2A) Suppose $t_2 < 1$. Then, the above inequality implies $W(t_0+2T) \leq 8K_{wz}(exp[-g_2(2T-1)] + \frac{2K_2}{g_2L})^2 W(t_0+t_2)$. Using Lemma 9.3(iv), we have $W(t_2+t_0) \leq 8K_{wz}exp[2(K_z+\epsilon_z)]W(t_0)$, which implies that $W(t_0+2T) \leq 64K_{wz}^2 exp[2(K_z+\epsilon_z)](exp[-g_2(2T-1)] + \frac{2K_2}{g_2L})^2 W(t_0)$, so that $W(t_0+2T) \leq \gamma W(t_0)$, the last inequality following from the definitions of T and L.

(2B) Suppose $t_2 \geq 1$. In this case, we have by Lemma 9.3(iii) and the definitions of T and L, $W(t_0+t_2) \leq exp[-\beta t_2/4](k+\epsilon_0)W(t_0)$, so that $W(t_0+2T) \leq 8exp[-\beta/4](k+\epsilon_0)K_{wz}(exp[-g_2T] + \frac{2K_2}{g_2L})^2 W(t_0) \leq \gamma W(t_0)$.

Since the rest of the proof is similar, we abbreviate it.

Case 3 Suppose $0 \leq t_1 \leq T \leq t_2 < 2T$. We apply the definitions of T and L to the following cases just as in cases 1 and 2 above and in each case get $W(t_0+2T) \leq \gamma W(t_0)$.

(3A) Suppose $I(t_0+T) = 1$. Then we separately consider the case where $I(t_0+t) = 1$, for some $t \in [t_2, t_2+T]$, and where $I(t_0+t) = 0, \forall t \in [t_2, t_2+T]$.

(3B) Suppose $I(t_0+T) = 0$. Define $t_3 := \max\{t \in [0,T] : I(t_0+t) = 1\}$.

(1) $(t_3 < 1)$

(a) $(t_2 - T < 1)$ (i) $(I(t_0+t) = 1,\ \text{for some } t \in [t_2, t_2+T])$

(ii) $(I(t_0+t) = 0, \forall t \in [t_2, t_2+T])$

(b) $(t_2 - T \geq 1)$ (i) $(I(t_0+t) = 1,\ \text{for some } t \in [t_2, t_2+T])$

(ii) $(I(t_0+t) = 0, \forall t \in [t_2, t_2+T])$.

(2) $(t_3 \geq 1)$

(a) $(t_2 - T < 1)$ (i) $(I(t_0+t) = 1,\ \text{for some } t \in [t_2, t_2+T])$

(ii) $(I(t_0+t) = 0, \forall t \in [t_2, t_2+T])$

(b) $(t_2 - T \geq 1)$ (i) $(I(t_0+t) = 1,\ \text{for some } t \in [t_2, t_2+T])$

(ii) $(I(t_0+t) = 0, \forall t \in [t_2, t_2+T])$.

Here one needs to consider the four cases $(t_3 < T/2)/(t_3 \geq T/2)$ and $(t_2 - T < T/2)/(t_2 - T \geq T/2)$. (Because of symmetry, these reduce to just two).

Case 4 $(T \leq t_1 \leq t_2 < 2T)$. Again, we apply the definitions of T and L as in cases 1 and 2 to get $W(t_0 + 2T) \leq \gamma W(t_0)$ in each case.

(a) $(I(t_0 + t) = 0\,, \forall t \in [t_2, t_2 + T])$ (i)$(t_2 - T < 1)$ (ii)$(t_2 - T \geq 1)$

(b) $(I(t_0 + t) = 1\,,$ for some $t \in [t_2, t_2 + T])$.

□

Proof of Theorem 9.1. Lemma 9.4 and the fact that z has a bounded growth rate gives us the desired result since $W(t + 2T) \leq 8K_{wz}exp[4(K_z + \epsilon_z)T]W(t)$ whenever $W(\tau) \geq K_{wz}L^2, \forall \tau \in [t, t + 2T]$. Both K_{wz} and L are independent of initial conditions. However, T_{large} can depend on initial conditions. This concludes the proof. □

Theorem 9.2. *(i)* $\|\psi\|, \|\psi'\| \in L_\infty$*, (ii)* $v, v_f, v'_f \in L_\infty$*, (iii)* $y \in L_\infty$*, (iv)* $u \in L_\infty$*,* (v) $e_a \in L_\infty$*, (vi)* $\zeta \in L_\infty$*, and (vii)* $\dot{\tilde{\theta}} \in L_\infty$*.*

Proof. The result (i) follows from Theorem 5.1(iii). The result (ii) follows from assumption (A4) and Theorem 5.1(i),(ii). The result (iii) follows since $y = \psi^T\theta + v'_f + \xi_{2d_0}$, and $\|\theta\| \leq M$. Lemma 5.1 implies that (iv) is true, and since $e_a = -\psi^T\tilde{\theta} + v'_f + \xi_{2d_0}$, and $\|\tilde{\theta}\| \leq \tilde{M}$, (v) holds true. The result (vi) follows since $\zeta = h^T exp[Q_1 t] * (H^T\dot{\tilde{\theta}})$, and $\|H^T\dot{\tilde{\theta}}\| \leq C\alpha e_a$. Finally, result (vii) follows from property (P1) of the parameter estimator. □

10 Performance for a Nominal System

We now show that if the system has no unmodeled dynamics and disturbances, then exact asymptotic tracking is achieved.

Theorem 10.1. *If* $K_v = 0, k_v = 0$*, and* $\mu_m = 0$*, then* $|y(t) - y_m(t)| \to 0$ *as* $t \to \infty$*.*

Proof. It suffices to show $e_a(t) \to 0$ and $\zeta(t) \to 0$ as $t \to \infty$, since $e_1(t) = e_a(t) - \zeta(t)$. Note that since $K_v = 0, k_v = 0, \mu_m = 0$ we have $v_f'(t) \equiv 0$. So, by Lemma A.2, $\frac{e_a^2}{n} \leq -\frac{\dot{V}}{\alpha} + cexp[-2d_0 t]$. This implies, since $\|\tilde{\theta}\| \leq \tilde{M}$, that

$$\int_0^\infty \frac{e_a^2(\tau)}{n(\tau)} d\tau \leq \frac{\tilde{M}^2}{\alpha} + \frac{k}{2d_0}. \tag{35}$$

Boundedness of $n(\cdot)$ then implies $e_a \in L_2$. Recall that $e_a \in L_\infty$. $e_a = -\tilde{\theta}^T \psi + \xi_{2d_0}$ implies $|\dot{e_a}| \leq \|\dot{\tilde{\theta}}\| \|\psi\| + \|\tilde{\theta}\| \|\dot{\psi}\| + c_0 exp[-2d_0 t]$, which implies $\dot{e_a} \in L_\infty$. Therefore, by Barbalat's Lemma [Popov [1] p.211, or Sastry & Bodson [17] p.19], $e_a(t) \to 0$ as $t \to \infty$. Now, recall that $\zeta(t) = h^T exp[Q_1 t] * (H(t)\dot{\tilde{\theta}}(t))$, and $\|H\dot{\tilde{\theta}}\| \leq \alpha C |e_a|$, which implies,

$$|\zeta(t)| \quad \leq \quad C\alpha \int_0^t exp[-a(t-\tau)/2] |e_a(\tau)| d\tau.$$

Using the just established fact that $e_a \to 0$, it follows that $\zeta \to 0$. □

11 Robust Performance

We now show that the performance of the adaptive tracker, as measured by the mean square tracking error, is robust in that it is a quadratic (hence also continuous) in the magnitude of the unmodeled dynamics and bounded disturbances.

Theorem 11.1.

$$\limsup_{T \to \infty} \frac{1}{T_2} \int_0^{T_2} (y(t) - y_m(t))^2 dt \leq c(K_v^2 + k_v^2 + \mu_m^2). \tag{36}$$

where c is a generic constant which can only decrease (or remain constant) as K_v, k_v, k_{v0} and μ_m decrease.

Proof. From Lemma A.2, we have $\frac{e_a^2 - v_f'^2}{n} \leq -\frac{\dot{V}}{\alpha} + c_0 exp[-2d_0 t]$, which implies $\frac{1}{T_2}\int_{T_l}^{T_2} \frac{e_a^2 - v_f'^2}{n} \leq \frac{kexp[-2d_0 T_l]}{T_2}$, $\forall T_2 \geq T_l$. Hence, $\limsup_{T_2 \to \infty} \frac{1}{T_2}\int_{T_l}^{T_2} \frac{e_a^2 - v_f'^2}{n} \leq 0$, i.e. $\limsup_{T_2 \to \infty} \frac{1}{T_2}\int_{T_l}^{T_2} \frac{e_a^2}{n} \leq \limsup_{T_2 \to \infty} \frac{1}{T_2}\int_{T_l}^{T_2} \frac{v_f'^2}{n}$. Noting that $z(t) \geq \min\{z(0), K_2/g_2\} =: z_{min}$, and defining $z_{max} := \sup_{t \geq T_l} z(t)$, by (16), we get, $\frac{v_f'^2}{z^2} \leq 2(K_{vf}' K_{mz})^2 + 2\frac{(K_{vf}' k_{mz} + k_{vf} + k(T_l))^2}{z^2}$, which then gives $\frac{1}{T_2}\int_{T_l}^{T_2} \frac{v_f'^2}{n} \leq [2(K_{vf}' K_{mz})^2 +$

$2\frac{(K'_{vf}k_{mz}+k_{vf}+k(T_l))^2}{z_{min}^2}]\frac{z_{max}^2}{\nu}$, $\forall T_2 \geq T_l$. This implies $\lim\sup_{T_2\to\infty} \frac{1}{T_2}\int_{T_l}^{T_2} \frac{e_a^2}{n} \leq c(K_v^2 + k_v^2 + \mu_m^2 + k^2(T_l))$, i.e. $\lim\sup_{T_2\to\infty} \frac{1}{T_2}\int_{T_l}^{T_2} e_a^2 \leq c(K_v^2 + k_v^2 + \mu_m^2 + k^2(T_l))$, where we recall that $k(T_l)$ depends only on initial conditions and decays exponentially at a rate faster than $exp[-2d_0T_l]$ as T_l increases. Finally, using the fact that e_a is bounded and the fact that the expression above is true for all $T_l > 0$, we get $\lim\sup_{T_2\to\infty} \frac{1}{T_2}\int_0^{T_2} e_a^2 \leq c(K_v^2 + k_v^2 + \mu_m^2)$. The proportionality constant c is basically $z_{max}n_{max}$ and is therefore $O(z_{max}^2)$. From the Robust Ultimate Boundedness Theorem, we know that it decreases as the unmodeled dynamics and bounded disturbance decrease. This means that the right-hand side in the expression above indeed goes to zero as the unmodeled dynamics and bounded disturbances go to zero, thereby giving us robust performance. Now consider the swapping term. Recall from the proof of Theorem 6.1 that,

$$\int_0^t \zeta^2(t')dt' \leq \quad (\alpha C)^2 \textstyle\int_0^t (\int_0^{t'} exp[-a(t'-\tau)/2]|e_a(\tau)|d\tau)^2 dt' \quad \leq \frac{8}{a^2}(\alpha C)^2 \int_0^t e_a^2(\tau)d\tau\,.$$

This implies that $\lim\sup_{T_2\to\infty} \frac{1}{T_2}\int_0^{T_2} \zeta^2(t')dt' \leq \frac{8}{a^2}(\alpha C)^2(K_v^2 + k_v^2 + \mu_m^2)c$. Finally, recalling that $e_1(t) = e_a(t) - \zeta(t)$ yields the desired result. □

12 A Simulation Example

We now present a simulation example to illustrate the results. Additional examples can be found in [20]. This example is the same as the one considered in [12], except that we *do not* assume knowledge of the (nominal) plant gain, and that we also add a bounded disturbance to the output. The actual plant is unstable and has a fast, nonminimum phase zero. This plant is modeled as a nominal plant which is minimum-phase and unstable, with a multiplicative uncertainty.

Example 1.

The true system is given by

$$y(t) = \frac{(1-\mu s)}{s(s-1)}u(t) + w(t) \tag{37}$$

where $0 < \mu < 1$ and $w(t)$ is a bounded disturbance. The reference model to be matched is

$$y_m(t) = \frac{1}{(s+1)(s+2)}r(t) \tag{38}$$

where $r(t) = 10sin(0.5t)$. We consider a nominal model of the form $y(t) = \frac{k}{(s+a)(s+b)}u(t)$, which is parameterized using $\lambda_1(s) = (s+1), \lambda_2(s) = (s+1)^2$ (so that $\lambda(s) = (s+1)^3$), which results in $F(s) = (s+4)$, $G(s) = (7s+1)$, and $\theta = (-6, 7, 3, 1)^T$. The simulation results with $\hat{\theta}(0) = (-4, 3, 6, 4)^T$, $\mu = 0.02$, $w(t) =$ unit square wave with period 10, parameter estimator constants $\alpha = 4.0, \nu = 1$ and $b_{min} = 0.5$, and constants $d_0 = 0.7, d_1 = 1.0, m(0) = 2.0$ for the overbounding signal $m(t)$ are given in Fig. 1. Despite the presence of unmodeled dynamics (and a particularly nasty-looking filtered bounded disturbance), the plant output approximately tracks the given reference model output over the time interval considered.

13 Concluding Remarks

In this paper, we have obtained boundedness and performance for continuous-time plants of *arbitrary* relative degree, with a somewhat wider class of unmodeled dynamics than in [12], but without any extra modifications except projection. Unlike [14], we allow non-differentiable bounded disturbances, and non-differentiable reference inputs. We also allow some time-varying and nonlinear uncertainties. The nominal plant is however restricted to be minimum-phase.

We have shown that eventually all the signals enter a closed, compact set, the size of which is independent of initial conditions. Also, the upper-bounds on the size of allowable unmodeled dynamics are independent of initial conditions.

Our results thus show that the projection mechanism alone is sufficient to guarantee robust boundedness and robust performance at least with respect to small unmodeled dynamics and bounded disturbance. It is important to study the dependence of the bounds on the parameters of the nominal plant, the constants defining the unmodeled dynamics, initial conditions, etc. Also, it is important to reevaluate the various robustness modifications which have earlier been proposed, to examine the amount of robustness they provide, the performance guaranteed in the presence of unmodeled dynamics and disturbances, and to thus determine whether they actually provide some improvements with respect to employing just the projection mechanism.

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Appendix A

Lemma A.1.

Consider the system: $\dot{x} = A_w x + b_w w_{in}, w_{out} = h_w^T x$ with zero initial conditions. (A_w, b_w, h_w^T) is a minimal representation of $H_0(s) = \frac{B'(s)}{A(s)}$, where $H_0(s)$ is strictly proper and $B'(s - p_0)$ is Hurwitz.

If $|w_{out}| \leq K_o m + k_o + c_0 exp[-pt]$, then, $\|x\| \leq K_x m + k_x + c_0 exp[-pt]$ for some positive constants K_x and k_x.

Proof. Without loss of generality, suppose

$$H_0(s) = \frac{B'(s)}{\prod_{i=1}^{k}(s^2 + a_{i1}s + a_{i2})\prod_{j=1}^{n-2k}(s + a_j)}$$

Since $H_0(s)$ is minimal, the corresponding states are the states corresponding to $\frac{w_{in}}{s^2+a_{i1}s+a_{i2}}, i = 1, \ldots, k$ and $\frac{w_{in}}{s+a_j}, j = 1, \ldots, n-2k$. Now,

$$\frac{1}{s + a_l} w_{in}(t) = \frac{\prod_{i=1}^{k}(s^2 + a_{i1}s + a_{i2})\prod_{j=1, j\neq l}^{n-2k}(s + a_j)}{B'(s)} w_{out}(t)$$

Using Lemma 5.2(ii) (since $B'(s - p_0)$ is Hurwitz), we get $|\frac{1}{s+a_l} w_{in}(t)| \leq cm(t) + c + c_0 exp[-pt], l = 1, \ldots, n-2k$. Define $w_l(t) = \frac{1}{s^2+a_{l1}s+a_{l2}} w_{in}(t) =: H_l(s) w_{in}(t)$. Then, since

$$w_l(t) = \frac{\prod_{i=1, i\neq l}^{k}(s^2 + a_{i1}s + a_{i2})\prod_{j=1}^{n-2k}(s + a_j)}{B'(s)} w_{out}(t) =: H_{lo}(s) w_{out}(t),$$

using Lemma 5.2(i) (since $B'(s - p_0)$ is Hurwitz), we get $|w_l(t)| \leq cm(t) + c + c_0 exp[-pt], l = 1, \ldots, k$. Further, since $H_{lo}(s)$ is strictly proper, Lemma 5.2(ii) gives $|\dot{w}_l(t)| = |sH_{lo}(s) w_{out}(t)| \leq cm(t) + c + c_0 exp[-pt]$. Since $w_l(t)$ and $\dot{w}_l(t)$ are the states corresponding to $H_l(s) w_{in}(t)$, we are done. □

Lemma A.2. *For the parameter estimator with projection, define* $V(t) := \|\tilde{\theta}(t)\|^2$. *Then, we have*

$$\dot{V}(t) \leq -\alpha \frac{(\tilde{\theta}^T(t)\psi(t))^2}{n(t)} + \alpha \frac{v_f'^2(t)}{n(t)} + \alpha \frac{c_0 \tilde{M}}{\sqrt{n(t)}} exp[-2d_0 t] \tag{39}$$

Furthermore, (39) also holds when the term $(\tilde{\theta}^T(t)\psi(t))$ *on the right-hand side is replaced by* $e_a(t)$.

Proof. For notational simplicity, let $c_1(t) := \frac{\hat{\theta}(t)}{||\hat{\theta}(t)||}$, and $c_2(t) := \frac{\alpha\psi(t)e_a(t)}{n(t)}$. First, consider the parameter estimator when the projection is not used, i.e. $\dot{\hat{\theta}}(t) = c_2(t)$. Then, recalling that $e_a(t) = -\tilde{\theta}^T(t)\psi(t) + v_f'(t) + \xi_{2d_0}(t)$, we obtain

$$\begin{aligned}
\dot{V}(t) &= 2\tilde{\theta}(t)^T \frac{\alpha\psi(t)e_a(t)}{n(t)} \\
&= \frac{\alpha}{n(t)}[-2(\tilde{\theta}^T(t)\psi(t))^2 + 2(\tilde{\theta}^T(t)\psi(t))v_f'(t) + 2(\tilde{\theta}^T(t)\psi(t))\xi_{2d_0}(t)] \\
&\leq \frac{\alpha}{n(t)}[-(\tilde{\theta}^T(t)\psi(t))^2 + v_f'^2(t)] + \alpha\frac{c_0\tilde{M}}{\sqrt{n(t)}}exp[-2d_0t].
\end{aligned}$$

Now, consider the parameter estimator with projection. Using property (P2) of the projection, we get,

$$\dot{V} = 2\tilde{\theta}^T\dot{\hat{\theta}} \;\; = \;\; 2\tilde{\theta}^T Proj(\hat{\theta}, c_2) \leq 2\tilde{\theta}^T c_2$$

and so the bound for the estimator without projection still holds. □

Appendix B

In this section, we will assume that (i) $t \geq T_l$, and (ii) $W(\cdot) \geq K_{wz}L^2$, which implies that $z(\cdot) \geq L/2$.

Lemma B.1.

(i) $\frac{||\psi||}{z}, \frac{||\phi_y||}{z}, \frac{||\phi_u||}{z}, \frac{||\phi||}{z} \leq c$.

(ii) $\frac{||\psi^{(i)}||}{z} \leq a_i$, $i = 1, \ldots, n-m$.

(iii) $|\frac{d}{dt}(y - v_f')| \leq k_y'z + k(T_l)$, *for some* $k_y' > 0$, *and* $k(T_l)$ *is a positive constant which decreases exponentially with increasing* T_l.

(iv) $|\frac{d}{dt}(u - r'/\hat{b}_m + \hat{\theta}_n v_f'/\hat{b}_m)| \leq k_u'z + k_{v\mu} + k(T_l)$, *for some* $k_u' > 0$, *where* $k(T_l)$ *is a positive constant which decreases exponentially with increasing* T_l *and* $k_{v\mu}$ *is a positive constant which is a weighted linear combination of* K_v, k_v *and* μ_m.

Proof. The result (i) is immediate from Theorems 5.1(iii),(v),(vi), and 6.1(ii), while (ii) follows since $\psi^T = (\frac{y}{(s+a)^{2n-m-1}}, \ldots, \frac{y}{(s+a)^{n-m}}, \frac{u}{(s+a)^{2n-m-1}}, \ldots, \frac{u}{(s+a)^{n-m}})$. For (iii) note that $y - v_f' = \psi^T\theta + \xi_{2d_0}$, so that by (ii) above $|\frac{d}{dt}(y - v_f')| \leq a_1Mz + k(T_l)$. For (iv) recall that the control law is $u - \frac{1}{\hat{b}_m}r' + \frac{\hat{\theta}_n}{\hat{b}_m}v_f' = -\frac{1}{\hat{b}_m}[\hat{\theta}_u^T\phi_u + \hat{\theta}_y^T\phi_y + \hat{\theta}_n(y - v_f')] =: RHE/\hat{b}_m$. This implies

$$\begin{aligned}
\frac{d}{dt}(u - \frac{1}{\hat{b}_m}r' + \frac{\hat{\theta}_n}{\hat{b}_m}v_f') &= -\frac{\dot{\hat{b}}_m}{\hat{b}_m^2}(RHE) - \frac{1}{\hat{b}_m}(\dot{\hat{\theta}}_u^T\phi_u + \dot{\hat{\theta}}_y^T\phi_y + \\
&\qquad \hat{\theta}_u^T\dot{\phi}_u + \hat{\theta}_y^T\dot{\phi}_y + \dot{\hat{\theta}}_n(y - v_f') + \hat{\theta}_n\frac{d}{dt}(y - v_f'))
\end{aligned}$$

Now, $\|\dot{\hat{b}}_m \phi_u\| \leq |\dot{\hat{b}}_m| \|\phi_u\| \leq \|\dot{\hat{\theta}}\| \|\psi'\| \leq \frac{\alpha \|\psi'\|^2 |e_a|}{n} \leq \alpha |e_a| \leq cz + k_{v\mu} + k(T_l)$. Similar analysis using $\|\hat{\theta}\| \leq M$ and $b_m \geq b_{min} > 0$, gives $|\frac{\dot{\hat{b}}_m}{\hat{b}_m^2}(RHE)| \leq cz + k_{v\mu} + k(T_l)$. Also, $|\dot{\hat{\theta}}_u^T \phi_u| \leq cz + k_{v\mu} + k(T_l)$, etc. Finally, using these results, the fact that $\hat{\theta}$ is bounded, and (i), (ii), (iii), etc., we get the desired result. □

For future use, define

$$
\begin{aligned}
k_{nz}(T) &= \frac{1}{K_{nz}}, \text{ if } I(t) = 1; \\
&= 1/\delta(T), \text{ if } I(t) = 0, \text{ and } I(t + t') = 1 \text{ for some } t' \in (0, T].
\end{aligned}
$$

Lemma B.2. *For instants t such that $I(t + t') = 1$ for some $t' \in [0, T]$,*

$$
|\frac{d}{dt}(\frac{\tilde{\theta}^T \psi^{(i)}}{z})(t)| \leq k_i(T), \quad i = 0, \ldots, n - m - 1. \tag{40}
$$

Proof. Note that $|\frac{d}{dt}(\frac{\tilde{\theta}^T \psi^{(i)}}{z})| \leq \frac{\|\dot{\tilde{\theta}}\| \|\psi^{(i)}\|}{z} + \frac{\|\tilde{\theta}\| \|\psi^{(i+1)}\|}{z} + \frac{\|\tilde{\theta}\| \|\psi^{(i)}\|}{z} \frac{|\dot{z}|}{z}$. Now, the parameter-update law yields $\|\dot{\hat{\theta}}\| \leq \frac{\alpha \|\psi\| (|\tilde{\theta}^T \psi| + |v'_f|)}{n} \frac{z^2}{z^2}$ and, $\frac{z^2(t)}{n(t)} \leq k_{nz}(T)$. This implies that $\|\dot{\hat{\theta}}\| \leq k(T)$. Finally, using (16), (22) and Lemma B.1(ii), we get the desired result. □

Lemma B.3. *For instants t such that $I(t + t') = 1$ for some $t' \in [0, T]$,*

$$
|\frac{d}{dt}(\frac{\tilde{\theta}^T \psi^{(n-m)} - (1 - \frac{\theta_{2n}}{\hat{b}_m})r' + (\theta_n - \theta_{2n}\frac{\hat{\theta}_n}{\hat{b}_m})v'_f}{z})| \leq k_{n-m}(T) \tag{41}
$$

Proof. Since $\phi = (s + a)^{(n-m)}\psi$, we have $\psi^{(n-m)} = \phi - \sum_{i=0}^{n-m-1} c_i \psi^{(i)}$, where c_i are the appropriate constants which depend on a. Using the control law (6), we get,

$$
\begin{aligned}
\tilde{\theta}^T \phi &= \hat{\theta}^T \phi - \theta^T \phi = r' - \theta^T \phi \\
&= r' - (\theta_u^T \phi_u + \theta_y^T \phi_y) - \theta_n (y - v'_f) - \theta_{2n}(u - \frac{1}{\hat{b}_m} r' + \frac{\hat{\theta}_n}{\hat{b}_m} v'_f) - \theta_n v'_f - \frac{\theta_{2n}}{\hat{b}_m} r' + \theta_{2n} \frac{\hat{\theta}_n}{\hat{b}_m} v'_f
\end{aligned}
$$

which implies

$$
\begin{aligned}
\phi^T \tilde{\theta} - (1 - \frac{\theta_{2n}}{\hat{b}_m})r' + (\theta_n - \theta_{2n}\frac{\hat{\theta}_n}{\hat{b}_m})v'_f &= -(\theta_u^T \phi_u + \theta_y^T \phi_y) - \theta_n (y - v'_f) \\
&\quad -\theta_{2n}(u - \frac{1}{\hat{b}_m} r' + \frac{\hat{\theta}_n}{\hat{b}_m} v'_f)
\end{aligned}
$$

Applying Lemmas B.1, B.2 to the above equation yields the desired result, using (22). □

Let $T' \leq T$. Assume that $I(t+t') = 1$ for some $t' \in [T', T]$. Then, from Lemma A.2, we have:

$$\begin{aligned}
&\frac{1}{T'}\int_t^{t+T'} \frac{(\tilde{\theta}^T\psi)^2}{z^2}d\tau \\
\leq\ & \sup \frac{n}{z^2}[\frac{V(t)-V(t+T')}{\alpha T'} + \frac{1}{T'}\sup\frac{z^2}{n}\int_t^{t+T'}(\frac{v_f'}{z})^2 d\tau + \frac{\alpha c_0\tilde{M}}{2\nu T' d_0}exp[-2d_0 t]] \\
\leq\ & \frac{1}{T'}(\frac{K_{nn}M^2}{\alpha}) + K_{nn}(K'_{vf}K_{mz} + \frac{K'_{vf}k_{mz}+k_{vf}+k(T_l)}{L/2})^2 k_{nz}(T) + \frac{\alpha c_0\tilde{M}}{2\nu T' d_0}exp[-2d_0 T_l]
\end{aligned}$$

where the supremum above is taken over the interval $[t, t+T']$. Defining $\mu := K'_{vf}K_{mz} + \frac{K'_{vf}k_{mz}+k_{vf}+k(T_l)}{L/2}$, the above inequality becomes,

$$\frac{1}{T'}\int_t^{t+T'} \frac{(\tilde{\theta}^T\psi)^2}{z^2}d\tau \quad \leq \quad \frac{1}{T'}((\frac{K_{nn}M^2}{\alpha}) + \frac{\alpha c_0\tilde{M}}{2\nu d_0}exp[-2d_0T_l]) + \mu^2 K_{nn}k_{nz}(T)$$

Subsequently we will apply Lemma B.4 to this inequality. The important point to note is that μ can be made as small (but positive) as we please by making K'_{vf} small enough and L large enough, and K'_{vf} can in turn be made as small as desired by making K_v and μ_m appropriately small.

Remark. *The constraint $1 \leq T'$ in Lemma B.4 stems from the fact that these depend on Theorem C1 of [12], which needs T' to be greater than or equal to one. These and all other details are provided in [20].*

Lemma B.4.

(i) *If $1 \leq T' \leq T$, and*

$$\frac{1}{T'}\int_t^{t+T'}(\frac{|\tilde{\theta}^T\psi|}{z})^2 \quad \leq \quad \frac{c_1}{T'} + \mu^2 c_2(T),$$

$$\textit{then,} \int_t^{t+T'}\frac{|\tilde{\theta}^T\phi|}{z} \quad \leq \quad \frac{\nu_3}{\epsilon^2} + [\mu^2\frac{\nu_4(T)}{\epsilon^2} + \mu\frac{\nu_5(T)}{\sqrt{\epsilon}} + \nu_6(T)\epsilon^\rho + 4a^{n-m}k(\mu,L)]T'$$

where $\epsilon \in (0,1]$, $\rho = 2^{-(n-m+1)}$, and $k(\mu, L) = \frac{(\frac{|\theta_{2n}|}{b_{min}}+1)k_r}{L/2} + (|\theta_n| + \frac{|\theta_{2n}|}{b_{min}}M)\mu$.

(ii) *If $1 \leq T' \leq T$, and*

$$\frac{1}{T'}\int_t^{t+T'}(\frac{|\tilde{\theta}^T\psi|}{z})^2 \quad \leq \quad \frac{c_1}{T} + \mu^2 c_2(T),$$

$$\textit{then,} \int_t^{t+T'}\frac{|\tilde{\theta}^T\psi|}{z} \quad \leq \quad \frac{\nu_7}{\epsilon^2} + [\mu^2\frac{\nu_8(T)}{\epsilon^2} + \sqrt{\epsilon}]T'$$

where $\epsilon \in (0,1]$.

Proof. The result (i) is an extension of the proof of Theorem C1 of [12], which uses Lemmas B.1, B.2 and B.3. Details of the proof are provided in [20]. The result (ii) is a special case of the proof of (i). $\square$

Table C

$K_{mz} = 1 + \frac{2C\alpha}{a-2d_0}$	$k_{mz} = \frac{d_1 k_{ym}}{d_0}$
$K_{u\psi'} = \frac{M(1+M)}{b_{min}}$	$K_{uv} = \frac{M}{b_{min}}$
$k_{uu} = \frac{k_r}{b_{min}}$	$K_{vf} = K_v c$
$k_{vf} = k_v c$	$k_{vf0} = k_{v0} c$
$K'_{vf} = K_{vf} + \mu_m c$	$K_{ym} = MK_{\psi' m} + K'_{vf}$
$K_{um} = K_{u\psi} K_{\psi' m} + K_{uv} K'_{vf}$	$k_{um} = K_{uv} k_{vf} + k_{uu}$
$k_u = k_{uu} + K_{uv}(K'_{vf} k_{mz} + k_{vf})$	$K_m = d_1(K_{um} + K_{ym}) - d_0$
$k_m = d_1(1 + k_{vf} + k_{um}$	$K_z = d_1 K_{mz}(K_{um} + \tilde{M} K_{\psi' m} + K'_{vf})$
$k_z = d_1\{[(\tilde{M} K_{\psi' m} + K'_{vf}) + K_{um}]k_{mz}$	$-(d_0 + g_2)$
$+k_{vf} + k_{um} + 1\}$	$k''_z = k_z + k(T_l)$
$\bar{d}_0 = \frac{d_0}{d_1} - K'_{vf} K_{uv} K_{mz}$	$\bar{g}_2 = \frac{g_2}{d_1}$
$\bar{K}_2 = \frac{K_2}{d_1} - 1 - k - k(T_l)$	$K_{nz} = \frac{(\bar{d}_0 - \bar{g}_2)^2}{4} \min\{\frac{1}{K^2_{u\psi'}}, \frac{1}{4\tilde{M}^2}\}$
$K_a = [2(1 - a + g_2) + 3(M^2 + K^2_{u\psi'})]$	$K_b = 3K'^2_{vf}(1 + K^2_{uv})K^2_{mz}$
$K_c = \max\{3[(K'_{vf} k_{mz} + k_{vf} + k(T_l))^2$	$K_d = K_b + \frac{4K_c}{L^2}$
$+(k_u + k(T_l))^2] - \nu K_a + 2g_2\nu, 0\}$	
$\delta(T) = exp[-K_a T][K_{nz} + \frac{K_d}{K_a}] - \frac{K_d}{K_a}$	$K_{wz} = \frac{1}{2} + 2\lambda_{max}(P)(C + 2g_4)^2$
$K_{yz} = (K'_{vf} + MK_{\psi' m})K_{mz}$	$K_{uz} = (K'_{vf} K_{uv} + K_{u\psi'} K_{\psi' m})K_{mz}$
$k_{yz} = MK_{\psi' m} k_{mz} + K'_{vf} k_{mz} + k_{vf}$	$k_{uz} = k_u + K_{u\psi'} K_{\psi' m} k_{mz}$
$K_{nn} = K^2_{\psi' m}(K_{mz} + \frac{2k_{mz}}{L})^2 + \frac{4\nu}{L^2}$	$\beta = \min\{\frac{1}{2\|P\|}, d_0 + g_2\}$
$\beta_1 = d_1 C$	$k_e = 2(\frac{4}{d_0+g_2}\beta_1^2 + \frac{d_0+g_2}{4})$

$$
\begin{aligned}
T \;=\; \max\Bigg\{ & 4, 1 + \frac{1}{g_2}\log\left(2\sqrt{\frac{64K^2_{Wz}\exp[2(K_z+\epsilon_z)](k+\epsilon_0)\exp[-\beta/4]}{\gamma}}\right), \\
& 1 + \frac{1}{g_2}\log\left(2\sqrt{\frac{1024K^3_{Wz}\exp[2(K_z+\epsilon_z)](k+\epsilon_0)\exp[-\beta/4]}{\gamma}}\right), \\
& \frac{1}{2g_2}\log\left(2\sqrt{\frac{8K_{Wz}}{\gamma}}\right), \frac{1}{g_2}\log\left(2\sqrt{\frac{1024K^3_{Wz}\exp[2(K_z+\epsilon_z)]}{\gamma}}\right), \\
& \frac{2}{\beta}\log\left(\frac{k+\epsilon_0}{\gamma}\right), \frac{1}{g_2}\log\left(2\sqrt{\frac{256K^2_{Wz}(k+\epsilon_0)\exp[-\beta/4]}{\gamma}}\right), \\
& \frac{1}{2}\left[1 + \frac{1}{g_2}\log\left(2\sqrt{\frac{64K^2_{Wz}\exp[2(K_z+\epsilon_z)]}{\gamma}}\right)\right], \frac{8}{\beta}\log\left(\frac{32K_{Wz}(k+\epsilon_0)}{\sqrt{\gamma}}\right),
\end{aligned}
$$

$$
\begin{aligned}
& \frac{1}{g_2}\log\left(2\sqrt{\frac{8K_{Wz}(k+\epsilon_0)\exp[-\beta/4]}{\gamma}}\right), \frac{2}{g_2}\log\left(2\sqrt{\frac{8K_{Wz}(k+\epsilon_0)}{\sqrt{\gamma}}}\right), \\
& \frac{4}{\beta}\log\left(\frac{256K_{Wz}^2(k+\epsilon_0)^2\exp[2(K_z+\epsilon_z)]}{\gamma}\right) - 1, \frac{4}{\beta}\log\left(\frac{32K_{Wz}(k+\epsilon_0)}{\gamma}\right), \\
& 1+\frac{1}{g_2}\log\left(2\sqrt{\frac{256K_{Wz}^3\exp[4(K_z+\epsilon_z)](k+\epsilon_0)\exp[-\beta/4]}{\gamma}}\right), \\
& 1+\frac{1}{g_2}\log\left(2\sqrt[4]{\frac{4096K_{Wz}^4\exp[4(K_z+\epsilon_z)]}{\gamma}}\right)\Bigg\}, \\
\epsilon &= \min\left\{\sqrt[1/\rho]{\frac{\beta}{8\beta_3\nu_6(T)}}, \left(\frac{\beta}{8\beta_5}\right)^2\right\}, \\
L &= \max\Bigg\{\frac{2}{K_z}(k_z+k(T_\ell)), \frac{2K_2}{2K_z-g_2}, \frac{2k_z''}{\epsilon_z}, \frac{4K_2}{g_2}\sqrt{\frac{64K_{Wz}^2\exp[2(K_z+\epsilon_z)]}{\gamma}}, \\
& \frac{4K_2}{g_2}\sqrt{\frac{8K_{Wz}(k+\epsilon_0)\exp[-\beta/4]}{\gamma}}, \sqrt{\frac{\beta_4 k\exp[2\beta T]}{\beta K_{Wz}\epsilon_0}}, \frac{4K_2}{g_2}\sqrt{\frac{8K_{Wz}(k+\epsilon_0)}{\sqrt{\gamma}}}, \\
& \frac{2K_2}{g_2}, \frac{4K_2}{g_2}\sqrt{\frac{256K_{Wz}^3\exp[4(K_z+\epsilon_z)](k+\epsilon_0)\exp[-\beta/4]}{\gamma}}, \\
& \frac{4K_2}{g_2}\sqrt{\frac{8K_{Wz}}{\gamma}}, \frac{4K_2}{g_2}\sqrt{\frac{64K_{Wz}^2\exp[2(K_z+\epsilon_z)](k+\epsilon_0)\exp[-\beta/4]}{\gamma}}, \\
& \frac{4K_2}{g_2}\sqrt{\frac{4096K_{Wz}^4\exp[4(K_z+\epsilon_z)]}{\gamma}}, \sqrt{\frac{4k_{Wz}}{3K_{Wz}}}, \\
& \frac{4K_2}{g_2}\sqrt{\frac{1024K_{Wz}^3\exp[2(K_z+\epsilon_z)](k+\epsilon_0)\exp[-\beta/4]}{\gamma}}\Bigg\}.
\end{aligned}
$$

Increase L (if necessary) and choose T_l, K_2 large enough and K_{vmax}, μ_{max} small enough so that,

$(\bar{d}_0 - \bar{g}_2) \geq 4K'_{vf}K_{mz}$,

$\bar{K}_2 \geq 2(K'_{vf}k_{mz} + k_{vf} + k(T_l))$,

$\delta(T) \geq k$, for some positive scalar k,

$k(\mu_{max}, L) \leq \frac{\beta}{32\beta_3 a^{n-m}}$,

$\mu_1 \leq \sqrt{\frac{d_0+g_2}{8k_e}}$,

$\mu_2 \leq \frac{d_0+g_2}{8}$, and

$\mu_{max} \leq \min\{\frac{\beta\sqrt{\epsilon}}{8\beta_3\nu_5(T)}, \epsilon\sqrt{\frac{\beta}{8\beta_3\nu_4(T)}}, \epsilon\sqrt{\frac{\beta}{8\beta_5\nu_8(T)}}\}$,

where,

$\mu := K'_{vf}K_{mz} + \frac{K'_{vf}k_{mz}+k_{vf}+k(T_l)}{L/2}$,

$k(\mu, L) := \frac{k_r(1+\frac{|\theta_{2n}|}{b_{min}})}{L/2} + (|\theta_n| + \frac{|\theta_{2n}|}{b_{min}}M)\mu$,

$\mu_1 := K_v\gamma_2K_{mz} + \mu_m\gamma_3 c$, and

$\mu_2 := (K_vC + K'_{vf})d_1K_{mz}$.

Please refer to [20] for details.