SELF-TUNING TRACKERS

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Abstract

We examine the problem of obtaining adaptive control laws which tune themselves to control laws minimizing the variance of the tracking error between the output of the linear ARMAX system and a specified reference trajectory. If the reference trajectory is sufficiently rich of order greater than or equal to the sum of the degrees of the control and noise polynomials in the ARMAX system, then an adaptive controller is exhibited for which the parameter estimates are strongly consistent. For the linear model following problem where the trajectory to be tracked is generated as the output of linear system, it is enough for the order of sufficient richness to be greater than the degree of the noise polynomial alone. Further, if the order of sufficient richness is even smaller, as is often the case, then a lower dimensional adaptive controller which does not attempt to estimate all the coefficients of the noise polynomial is self-tuning.

Key words. adaptive control, self-tuning tracker

AMS(MOS) subject classifications. 93C40, 93E12, 93E20, 93C55

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1 INTRODUCTION

The problem of stochastic adaptive control of linear ARMAX systems has received considerable attention over the past decade. The notable pioneering contributions are due to Åström and Wittenmark [1] and Ljung [2,3]. Subsequently, Goodwin, Ramadge and Caines [4] and Goodwin and co-workers [5] have proved the *self-optimality* of some adaptive control algorithms for minimum variance regulation and tracking. By *self-optimality* it is meant that the cost, the time average of the square of the tracking error, is minimal.

Recently a stochastic gradient algorithm has been proved to be *self-tuning* for the *regulation* problem, see [6]. (Recall that in the regulation problem one wants the output of the system to stay as close as possible to zero, whereas in the tracking problem one wants to track a given arbitrary trajectory). By "self-tuning" it is meant that the adaptive control *law* converges to the optimal control law. This is clearly a property of fundamental interest since it implies that the adaptive controller can be used as a mechanism for tuning to the parameters of an optimal control law.

In this paper we examine the problem of minimum variance *tracking* where the goal is to ensure that the output of the system tracks a specified reference trajectory with minimal average squared tracking error.

From a purely technical viewpoint the analysis of the tracking problem along the lines of [6] has until now been stymied by the fact that a key geometric property of the adaptive control algorithm, which renders the regression and parameter estimate vectors orthogonal, holds only in the regulation problem and not in the case of tracking. Our first contribution here is to show how to overcome this difficulty by enlarging the dimension of the regression vector.

Another well known difficulty with the tracking problem is that when the reference trajectory to be tracked is a general non-zero trajectory (we call this the *general tracking problem*), then the control law which allows the trajectory to be tracked with minimum variance does require explicit knowledge of the coefficients of the colored noise polynomial, see [4], [6], [12]. This is another feature distinguishing the tracking problem from the regulation problem. Consequently, it is necessary to identify some additional parameters pertaining to the colored noise polynomial in order to obtain self-tuning. Such identification is established in this paper under the natural assumption that the reference trajectory is sufficiently rich of appropriate order.

The second essential contribution of this paper is the examination of how one may obtain self-tuning when the reference trajectory is not so rich as to allow one to identify all the coefficients of the colored noise polynomial. For example, in an important class of practical problems, called *set-point problems*, the output of the system is required to stay as close as possible to a certain specified level. Thus the reference trajectory is a non-zero constant, which is sufficiently rich of order one only. We examine such problems, which violate the richness assumptions of the general tracking problem, by examining the problem of following trajectories which are generated by linear models. We call these the *linear model following problems.* (The set-point problem is a special case of the linear model following problem). Our second class of main results is to show how one may adjust the *dimension* of the regression vector to the degree of excitation present in the reference trajectory. We then provide a proof of self-tuning of the resulting reduced dimension adaptive controllers.

Our main results are therefore the following:

(i) The adaptive control laws in both the general tracking problem as well as the linear model following problem are self-optimal, i.e., the average squared tracking error is minimal (Theorem 3).

(ii) In the general tracking problem, if the reference trajectory is sufficiently rich of order at least equal to the sum of the degrees of the control and noise polynomials in the ARMAX representation of the system, then the parameter estimates are strongly consistent, i.e., they converge to the true values almost surely (Theorems 6,7). This result also implies that the adaptive controller is self-tuning, i.e., the adaptive control law converges to the optimal control almost surely (Theorem 7).

(iii) For the parameter estimates to be strongly consistent in the linear model following problem it is enough for the order of sufficient richness of the reference trajectory to be equal to the degree of the noise polynomial alone (Theorems 6 and 7). This again implies self-tuning (Theorem 7).

(iv) Often, the degree of sufficient richness is even smaller than the degree of the noise polynomial (e.g., the set-point problem). In such linear model following problems, a lower dimensional adaptive controller can be used. This lower dimensional adaptiver controller is self-tuning (Theorem 7). The parameter estimates also converge (Theorem 6).

However, since no attempt is made at estimating all the coefficients of the noise polynomial, the parameter estimates do not converge to the true values (i.e., we are using a *direct* adaptive control law).

Some comments on the nature of these results in comparison with the results in *deterministic* adaptive control are useful. In deterministic adaptive control, where there is no noise in the system, one can asymptotically obtain zero tracking error. However in stochastic adaptive control there is noise and one wants to reject as much of the noise as possible. Clearly *optimal* noise rejection will depend critically on the knowledge of the correlations inherent in the possibly colored noise. This is where the central problem of estimating the colored noise coefficients enters into the stochastic adaptive control problem. Indeed, in the present paper, the need for richness in the reference trajectory is intimately related precisely to the need for estimating the model of the colored noise.

2 The Adaptive Control Laws

We consider the ARMAX system

$$y(t) = \sum_{i=1}^{p} a_i y(t-i) + \sum_{i=1}^{q} b_i u(t-i) + \sum_{i=1}^{s} c_i w(t-i) + w(t) \qquad (1)$$

where y, u and w are, respectively, the output, input and white noise. The parameters $(a_1, \ldots, a_p, b_1, \ldots, b_q, c_1, \ldots, c_s)$ are unknown. The goal is to design an adaptive control law which ensures that the output follows a given bounded reference trajectory $\{y^*(t)\}$ with minimal average squared tracking error, and such that the adaptive control law asymptotically self-tunes to the optimal control law. It is an added bonus if the true parameters $(a_1, \ldots, a_p, b_1, \ldots, b_q, c_1, \ldots, c_s)$ can also be asymptotically identified.

If the reference trajectory is arbitrary, we shall refer to this problem as the *general tracking problem*. In many problems however the reference trajectory is generated as the output of a *linear model*. We shall refer to such a special case as the *linear model following problem*. The special properties of a reference trajectory generated as the output of a linear model can be usefully exploited, as we will see in the sequel. We now discuss separately the general tracking problem and the linear model following problem.

2.1 The General Tracking Problem

In this case $\{y^*(t)\}$ is just a reference trajectory to be tracked with no special properties. We will use the following adaptive controller (with the notation $p \vee s := \max(p,s)$).

$$\theta(t+1) = \theta(t) + \frac{\mu\phi(t)}{r(t)}[y(t+1) - y^*(t+1)] \tag{2}$$

where, for the time being, $0 < \mu < 2$ is an arbitrary constant (but see the remark at the end of Section 4),

$$r(t+1) := 1 + \sum_{k=0}^{t+1} \phi^T(k)\phi(k), \tag{3}$$

$$\begin{aligned} \phi(t) &:= (y(t), \ldots, y(t-p\vee s+1), u(t), \ldots, u(t-q+1), \\ &\qquad -y^*(t+1), \ldots, -y^*(t-s+1)) \end{aligned} \tag{4}$$

$$u(t) := \frac{-1}{\beta_1(t)}\left[\sum_{i=1}^{p\vee s} \alpha_i(t) y(t-i+1) + \sum_{i=2}^{q} \beta_i(t) u(t-i+1) - \sum_{i=0}^{s} \gamma_i(t) y^*(t-i+1)\right] \tag{5}$$

where

$$(\alpha_1(t), \ldots, \alpha_{p\vee s}(t), \beta_1(t), \ldots, \beta_q(t), \gamma_0(t), \ldots, \gamma_s(t))^T := \theta(t). \tag{6}$$

Note that (5) can equivalently be written as

$$\phi^T(t)\theta(t) = 0. \tag{7}$$

The motivation behind this adaptive controller is the following. Rewrite the system (1) as,

$$\begin{aligned} y(t+1) - y^*(t+1) &= \left[\sum_{i=1}^{p} a_i y(t+1-i) + \sum_{i=1}^{q} b_i u(t+1-i) + \sum_{i=1}^{s} c_i w(t+1-i) - y^*(t+1)\right] \\ &\quad + w(t+1). \end{aligned}$$

If one could observe the past of $w(\cdot)$ at each time t, then an optimal controller would choose $u(t)$ so that the term in $[\cdot\cdot]$ on the right-hand side above is zero, i.e.,

$$u(t) = \frac{-1}{b_1}\left[\sum_{i=1}^{p} a_i y(t+1-i) + \sum_{i=2}^{q} b_i u(t+1-i) + \sum_{i=1}^{s} c_i w(t+1-i) - y^*(t+1)\right],$$

for this would result in $y(t+1) = y^*(t+1) + w(t+1)$, clearly yielding the best possible tracking error. However, the sequence $w(\cdot)$ is *not* observed, and so let us replace it by $y(\cdot) - y^*(\cdot)$, which is what we hope it would be, at least asymptotically. This gives the implementable control law,

$$u(t) = \frac{-1}{b_1}\left[\sum_{i=1}^{p\vee s}(a_i + c_i)y(t+1-i) + \sum_{i=2}^{q} b_i u(t+1-i) - \sum_{i=1}^{s} c_i y^*(t+1-i) - y^*(t+1)\right].$$

It can be shown that this control law is actually optimal with respect to the long run average of the square of the tracking error; for more details, see [12]. Let us define,

$$\theta^\circ := (a_1 + c_1, \ldots, a_{p\vee s} + c_{p\vee s}, b_1, \ldots, 1, c_1, \ldots, c_s)^T \tag{8}$$

(where, for convenience, we define $c_i := 0$ for $i > s$ and $a_i := 0$ for $i > p$ in (8)), and, under optimal control, the system (1) can be represented as

$$y(t+1) - y^*(t+1) = \phi^T(t)\theta^\circ + w(t+1),$$

while the optimal control law can be written as one which chooses $u(t)$ to satisfy,

$$\phi^T(t)\theta^\circ = 0.$$

Our adaptive control scheme (2)-(6) can be interpreted as trying to estimate θ° when the system is being optimally controlled.

Remark: Note that the $(p \vee s + q + 1)$th component of θ° is 1, and hence is a *known* quantity. *However, the estimator ignores this knowledge and estimates it anyway by $\gamma_0(t)$. We can therefore regard (2), (3) as an* unnormalized *parameter estimator. It follows that this parameter estimator is one dimension larger than that considered in Goodwin, Ramadge and Caines [4]. In this connection, it is also of interest to note that recently Wei [7] has proposed an estimator for the regulation problem which is one dimension less than [4], [6].*

2.2 The Linear Model Following Problem

In many situations of interest the reference trajectory is generated, at least asymptotically, as the output of a linear model. We shall suppose that there is a sequence $\{y_m(t)\}$ such that

$$y_m(t) = \sum_{i=1}^{l} h_i y_m(t-i) \tag{9}$$

and the trajectory to be tracked $y^*(t)$ is asymptotically close to $y_m(t)$ in that

$$\sum_{t=1}^{\infty}(y^*(t) - y_m(t))^2 < +\infty. \tag{10}$$

Without loss of generality we can make the following two assumptions:

There is no lower order difference equation satisfied by $\{y_m(t)\}$, i.e., there is *no* nontrivial polynomial $\overline{H}(z)$ of degree strictly less than l such that $\overline{H}(z)y_m(t) = 0$ for all t. (z is the *backward* shift operator).
(11a)

The roots of $H(z) := 1 - \sum_{i=1}^{l} h_i z_i$ are exactly on the unit circle and there are no repeated roots.
(11b)

Assumption (11a) is without loss of generality since otherwise we could simply replace $H(z)$ in (9) by $\overline{H}(z)$. Note that this also means that the initial conditions on (9) are sufficient to excite *all* the modes of $H(z)$. Assumption (11b) is also without loss of generality due to the following reasons. First, since we intend to work only with *bounded* $\{y^*(t)\}$, and since all the modes of $H(z)$ are excited, we have to assume that $H(z)$ has roots on or outside the unit circle, and also that the roots on the unit circle are not repeated. However, since we are only interested in the *asymptotic* behavior of $\{y^*(t)\}$, we can eliminate all the modes corresponding to roots of $H(z)$ which are strictly outside the unit circle, since they decay geometrically to 0. This leaves us with (11b).

It is worth noting that (11a) and (11b) together imply that

$$y_m(t) \quad = \quad d_0 + d_1(-1)^t + \sum d_i \sin(\omega_i t + \delta_i).$$

Depending on how large l is, we will use adaptive controllers with parameter estimators of different dimensions.

Case 1: $l \leq s$. Recall that s is the degree of the noise polynomial in (1). When $l \leq s$, we will reduce the dimension of the parameter estimator by $(s + 1 - l)$ components by replacing (4-6) by the following:

$$\begin{aligned} \phi(t) \quad &:= \quad (y(t), \ldots, y(t - p \vee s + 1), u(t), \ldots, u(t - q + 1), \\ &\qquad -y^*(t + 1), \ldots, -y^*(t + 2 - l))^T, & (12) \\ \theta(t) \quad &:= \quad (\alpha_1(t), \ldots, \alpha_{p \vee s}(t), \beta_1(t), \ldots, \beta_q(t), \gamma_0(t), \ldots, \gamma_{l-1}(t))^T, & (13) \end{aligned}$$

and

$$u(t) = \frac{-1}{\beta_1(t)}\left[\sum_{i=1}^{p\vee s}\alpha_i(t)y(t-i+1)+\sum_{i=2}^{q}\beta_i(t)u(t-i+1)-\sum_{i=0}^{l-1}\gamma_i(t)y^*(t-i+1)\right], \quad (14)$$

or equivalently by (7).

The idea underlying the above adaptive control law is the following. If the parameters were known the minimum variance adaptive control law would be,

$$u(t) = \frac{-1}{b_1}\left[\sum_{i=1}^{p\vee s}(a_i+c_i)y(t-i+1)+\sum_{i=2}^{q}b_iu(t-i+1)-y^*(t+1)-\sum_{i=1}^{s}c_iy^*(t-i+1)\right],$$

see [12] for details. In this control law the only terms featuring y^* are $y^*(t+1)+\sum_{i=1}^{s}y^*(t-i+1)=C(z)y^*(t+1)$. Thus the control law really only requires knowledge of $C(z)y^*(t)$. Let

$$G(z) := \sum_{i=0}^{l-1}g_iz^i \quad (15)$$

be a polynomial satisfying,

$$C(z) = F(z)H(z)+G(z) \quad (16)$$

for some

$$F(z) := \sum_{i=0}^{s-l}f_iz^i. \quad (17)$$

Such polynomials $G(z)$ and $F(z)$ are the remainder and quotient, respectively, when the polynomial $C(z)$ is divided by the polynomial $H(z)$. Then, asymptotically at least,

$$C(z)y^*(t) = [F(z)H(z)+G(z)]y^*(t) = F(z)H(z)y^*(t)+G(z)y^*(t) = G(z)y^*(t),$$

since by (9, 10), $H(z)y^*(t)=0$ holds asymptotically. Thus we only need knowledge of $G(z)y^*(t)$ in order to implement the true minimum variance control law. We can therefore interpret the parameter estimate (13) as trying to estimate

$$\theta^\circ := (a_1+c_1,\ldots,a_{p\vee s}+c_{p\vee s},b_1,\ldots,b_q,g_0,g_1,\ldots,g_{l-1})^T. \quad (18)$$

Remarks:

(i) The adaptive controller need not be provided with the precise information about what the polynomial $H(z)$ is. It only needs knowledge of the *degree* of $H(z)$.

(ii) It should be noted that the parameter estimator is no more "unnormalized," since the coefficients $g_0, \ldots, g_{l-1}$ are all unknown.

Case 2: $l \geq s+1$. Since $(s+1-l) \leq 0$ when $l \geq s+1$, no savings in dimensionality can be achieved. Hence we will use the same adaptive control law as (2-7). For this case also we define θ° as in (8).

3 Sufficient Richness

In the sequel we will prove that all the coefficients $(a_1, \ldots, a_p, b_1, \ldots, b_q, c_1, \ldots, c_s)$ can be asymptotically identified when the reference trajectory $\{y^*(t)\}$ is "sufficiently rich" in an appropriate sense. We have the following definition.

Definition (18). *We shall say that a scalar sequence $\{y^*(t)\}$ is strongly sufficiently rich of order l if l is the largest non-negative integer for which there exists an n and an $\epsilon > 0$ such that*

$$\sum_{k=t+1}^{t+n} (y^*(k-1), \ldots, y^*(k-l))^T (y^*(k-1), \ldots, y^*(k-l)) \quad \geq \quad \epsilon I_l \quad \text{forall} t \text{largeenough.}$$

I_l here is the $l \times l$ identity matrix.

The following property of $\{y_m(t)\}$, and also $\{y^*(t)\}$, generated by the linear model (9), (10), (11a), 11b) should be noted.

Lemma 1. *Suppose $\{y^*(t)\}$ and $\{y_m(t)\}$ satisfy (9)-(11b). Then both $\{y^*(t)\}$ and $\{y_m(t)\}$ are strongly sufficiently rich of order l.*

Proof: We will show that there exists $\epsilon > 0$ such that

$$\sum_{k=t+1}^{t+l} Y_l(k-1) \quad \geq \quad \epsilon I_l \text{forall} t \text{largeenough}$$

where

$$Y_l(k-1) \quad := \quad (y_m(k-1), \ldots, y_m(k-l))^T (y_m(k-1), \ldots, y_m(k-l)).$$

Suppose this is not true. Then there exists a sequence of vectors $\{x(t_n)\}$, with each $\|x(t_n)\| = 1$ and $x(t_n) =: (x_1(t_n), \ldots, x_l(t_n))^T$ such that

$$x^T(t_n) \sum_{k=t_n+1}^{t_n+l} Y_l(k-1)x(t_n) \quad \leq \quad \frac{1}{n}.$$

We can also assume without loss of generality that $\lim_n x(t_n) =: x$ exists with $\|x\| = 1$, $x := (x_1, \ldots, x_l)^T$. Moreover, since $\{y_m(t)\}$ is bounded, $\{Y_l(k-1)\}$ is also bounded and so

$$\lim_n x^T \sum_{k=t_n+1}^{t_n+l} Y_l(k-1)x \quad = \quad 0.$$

Let $X(z) := \sum_{i=1}^{l} x_i z^i$. Interpreting z as the *backward* shift operator, we have

$$\lim_n \sum_{k=t_n+1}^{t_n+l} [X(z)y_m(k)]^2 \quad = \quad 0.$$

This implies that

$$\lim_n X(z)y_m(t_n+i) = 0 \qquad \text{for} i = 1, 2, \ldots, l.$$

Now note that $H(z)X(z)y_m(t) = X(z)H(z)y_m(t) = 0$ and so

$$X(z)y_m(t) \quad = \quad \sum_{k=1}^{l} \delta_k \lambda_k^t$$

where $\{\lambda_k\}$ is the set of roots of $H(z)$. Hence we have

$$\lim_n \sum_{k=1}^{l} \delta_k \lambda_k^{t_n+i} = 0 \qquad \text{for} i = 1, \ldots, l.$$

This can also be written as

$$\lim_n \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & & & & \lambda_l \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_1^{l-1} & \lambda_2^{l-1} & & & \lambda_1^{l-1} & \end{bmatrix} \begin{bmatrix} \lambda_1^{t_n+1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2^{t_n+1} & & & 0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & & & \lambda_1^{t_n+1} & \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_l \end{bmatrix} \quad = \quad 0.$$

The first matrix on the left-hand side above is the Vandermonde matrix which is nonsingular since all the λ_k's are distinct. Moreover $|\lambda_k| = 1$ for all k, and so it follows that $\delta_k = 0$ for $k = 1, \ldots, l$. This however implies that $X(z)y_m(t) = 0$ for all t. However $X(z)$ is a polynomial of degree $l-1$ or less, and by (11a), it follows that $X(z) = 0$, i.e., $\|x\| = 0$. This is a contradiction to $\|x\| = 1$, proving that $\{y_m(t)\}$ is indeed strongly sufficiently rich of order l. By (10) it follows trivially that $\{y^*(t)\}$ is also strongly sufficiently rich of order l. (Actually it is enough that $\lim_t (y_m(t) - y^*(t)) = 0$).

For future reference, we also have the following result.

Lemma 2. *Let $S(t,z) := \sum_{i=0}^{j} s_i(t)z^i$. Suppose $\{s_i(t)\}$ is bounded for $i = 0, \ldots, j$ and $\lim_t |s_i(t) - s_i(t-1)| = 0$ for $i = 0, \ldots, j$. Suppose also that for some sequence $\{x(t)\}$,*

$$\lim_N \frac{1}{N} \sum_{t=1}^{N} [S(t,z)y_m(t)]^2 = 0 \quad \textit{and} \quad \lim_N \frac{1}{N} \sum_{t=1}^{N} x^2(t) = 0.$$

Then there exists a common subsequence $\{t_k\}$ with $\lim_k x(t_k) = 0$ and $\lim_k S(t_k, z) = K(z)H(z)$ for some polynomial $K(z)$. (By $S(t,z)y_m(t)$ we mean $\sum_{i=0}^{j} s_i(t)y_m(t-i)$.

Proof: Since $\lim_t |s_i(t) - s_i(t+n)| = 0$ for every n and $\{y_m(t)\}$ is bounded, it is also true that $\lim_N 1/N \sum_{t=1}^{N} [S(t+n,z)y_m(t)]^2 = 0$ for every n. Hence we can sum over n and also add $x^2(t)$ to get

$$\lim_N \frac{1}{N} \sum_{t=1}^{N} \left\{ x^2(t) + \sum_{n=1}^{l} [S(t,z)y_m(t-n)]^2 \right\} = 0.$$

Hence there is a subsequence $\{t_k\}$ such that

$$\lim_k S(t_k, z)y_m(t_k - n) = 0 \qquad \text{for} n = 1, \ldots, l \quad \lim_k x(t_k) = 0.$$

Further we can also assume without loss of generality that

$$\lim_k S(t_k, z) \quad =: \quad S(z)$$

exists, by which we mean that $\lim_k s_i(t_k) =: s_i$ exists for $i = 0, \ldots, j$ and $S(z) := \sum_{i=0}^{j} s_i z^i$. Further, since $\{y_m(t)\}$ is bounded, it follows that

$$\lim_k S(z)y_m(t_k - n) = 0 \qquad \text{for} n = 1, \ldots, l.$$

Note that $H(z)S(z)y_m(t) = 0$ for all t, and so

$$S(z)y_m(t) = \sum_{n=1}^{l} \delta_n \lambda_n^t$$

where $\{\lambda_n\}$ is the set of roots of $H(z)$. Proceeding just as in the proof of Lemma 1, it follows that

$$S(z)y_m(t) = 0 \qquad \text{forall} t.$$

Now let $U(z)$ be the greatest common divisor of $S(z)$ and $H(z)$. Then there exist polynomials $R(z)$ and $T(z)$ such that $R(z)S(z) + T(z)H(z) = U(z)$. Hence $U(z)y_m(t) = 0$ for all t. However, since the degree of $U(z)$ is less than or equal to l, it follows from (11a) that $U(z) = \xi H(z)$, for some scalar ξ, and so the Lemma is proved.

4 Assumptions

Define the polynomials

$$\begin{aligned} A(z) &:= 1 - \sum_{i=1}^{p} a_i z^i \\ B(z) &:= \sum_{i=1}^{q} b_i z^{i-1} \\ C(z) &:= 1 + \sum_{i=1}^{s} c_i z^i . \end{aligned}$$

Throughout this paper we employ the following assumptions only.

All the roots of $B(z)$ and $C(z)$ are strictly outside the unit circle. (19a)

$Re[C(e^{i\omega}) - \frac{\mu}{2}] 0 \text{for} 0 \leq \omega 2\pi$ (19b)

$b_1 ! = 0$ (19c)

$z^{-1}[C(z) - A(z)]$ and $B(z)$ are polynomials of degrees respectively equal to $(p \vee s - 1)$ and $(q - 1)$, which have no common factors. (19d)

$\{w(t)\}$ is a sequence of scalar random variables on a probability space $\{\Omega, F, P\}$, whose distributions are all mutually absolutely continuous with respect to Lebesgue measure. (19e)

Let $\{F_t := \sigma w(1), \ldots, w(t)\}$ be the sub-σ-algebra of F generated by $\{w(1), \ldots, w(t)\}$. We assume that there are $\sigma^2 0$ and $\delta 0$ such that (19f)

$E[w(t)|F_{t-1}] = 0 a.s.$

$E[w^2(t)|F_{t-1}] = \sigma^2 a.s.$

$\sup_t E[|w(t)|^{2+\delta}|F_{t-1}] + \infty a.s.$

$||\theta(0)||0$ (19g)

$\{y^*(t)\}$ is bounded. (19h)

It should be noted that the condition (19e) guarantees that the controls are well defined a.s. through (5.14) since the event $\{\beta_1(t) = 0\}$ is a null event, see Caines and Meyn [9].

Remark: Let us consider a different constant μ_1 in place of μ in (2). It is easy to verify, see [12], that the resulting adaptive control algorithm produces parameter estimates $\theta_1(t) = \mu_1/\mu\theta(t)$ and *identical* inputs and outputs as the original algorithm using μ, provided $\theta_1(0)$ is chosen as $\theta_1(0) := \mu_1/\mu\theta(0)$. This property relies on the fact that the control input $u(t)$ is invariant with respect to scaling of $\theta(t)$ in (7). Making use of this observation it follows that one need not restrict μ to lie in $(0, 2)$; it is enough to have $\mu \neq 0$. Further, one only needs the assumption

$$ReC(e^{i\omega}) 0 \text{for} 0 \leq \omega 2\pi \tag{19i}$$

in place of (19b).

5 Self-Optimality

In this section we will prove the following Theorem which asserts, among other things, that in all cases the adaptive controller minimizes the average squared tracking error.

Theorem 3.

$$\lim_N \frac{1}{N} \sum_{t=1}^{N} [y(t) - y*(t)]^2 = \sigma^2 a.s. \tag{20a}$$

$$\lim_N \frac{1}{N} \sum_{t=1}^{N} (E[y(t+1) - y*(t+1)|F_t])^2 = 0 a.s. \tag{20b}$$

$$\lim_N \frac{1}{N}\sum_{t=1}^{N} u^2(t) + \infty a.s. \tag{20c}$$

$$\lim_t ||\theta(t)-\theta^\circ||^2 \text{existsandisfinite} a.s. \tag{20d}$$

Proof: We will abbreviate those details of the proof which are similar to those of Goodwin, Ramadge and Caines [4] or [6]. Let $\widetilde{\theta}(t) := \theta(t) - \theta^\circ$ and define $V(t) := \|\widetilde{\theta}(t)\|^2$. Using $r(t) \geq \phi^T(t)\phi(t)$ and $\phi^T(t)\widetilde{\theta}(t) = -\phi^T(t)\theta^\circ$, we can get

$$\begin{aligned}
E[V(t+1)|F_t] \quad \leq \quad & V(t) - \frac{2\mu}{r(t)}\left\{\phi^T(t)\theta^\circ - \frac{\mu+\delta}{2}E[y(t+1)-y^*(t+1)|F_t]\right\} \\
& \cdot E[y(t+1) - y*(t+1)|F_t]\frac{-\mu\delta}{r(t)}(E[y(t+1)-y^*(t+1)|F^t])^2 \\
& +\mu^2\frac{\phi^T(t)\phi(t)}{r^2(t)}\sigma^2
\end{aligned}$$

for all δ. Choose $\delta > 0$ so small that $[C(z) - (\mu + \delta/2)]$ is strictly positive real. Let us first consider the following case.

Case 1: General tracking problem or the linear model following problem with $l \geq s+1$.

$$\begin{aligned}
C(z)E[y(t+1)-y^*(t+1)|F_t] \quad = \quad & C(z)[y(t+1)-y^*(t+1)-w(t+1)] \\
= \quad & [y(t+1)-y^*(t+1)-w(t+1)] \\
& +[C(z)-1][y(t+1)-y^*(t+1)-w(t+1)] \\
= \quad & [y(t+1)-y^*(t+1)-w(t+1)] \\
& +\textstyle\sum_{i=1}^{s} c_i[y(t-i+1)-y^*(t-i+1)-w(t-i+1)] \\
= \quad & [y(t+1)-w(t+1)-\textstyle\sum_{i=1}^{s} c_i w(t-i+1)] - y^*(t+1) \\
& +\textstyle\sum_{i=1}^{s} c_i[y(t-i+1)-y^*(t-i+1)] \\
= \quad & \textstyle\sum_{i=1}^{p\vee s}(a_i+c_i)y(t-i+1) + \sum_{i=1}^{q} b_i u(t-i+1) \\
& -y^*(t+1) - \textstyle\sum_{i=1}^{s} c_i y^*(t-i+1) \\
= \quad & \phi^T(t)\theta^\circ.
\end{aligned} \tag{21}$$

By the strict positive realness of $[C(z) - (\mu+\delta)/2]$ it therefore follows that

$$\begin{aligned}
S(n) \quad := \quad & 2\mu\sum_{t=1}^{n}\left\{\phi^T(t)\theta^\circ - \frac{\mu+\delta}{2}E[y(t+1)-y^*(t+1)|F_t]\right\}E[y(t+1)-y^*(t+1)|F_t] \\
\geq \quad & K\text{a.s. for all } n\text{, for some } K.
\end{aligned}$$

Defining $M(t) := V(t) + S(t-1)/r(t-1)$, and using $r(t) \geq r(t-1) > 0$, it follows that

$$E[M(t+1)|F_t] \;\; \leq \;\; M(t) - \frac{\mu\delta}{r(t)}(E[y(t+1) - y^*(t+1)|F_t])^2 + \frac{\mu^2\phi^T(t)\phi(t)}{r^2(t)}\sigma^2.$$

The last term above is summable a.s., and so using the Positive Near Supermartingale Convergence Theorem we can get:

(i) $\{M(t)\}$ converges a.s.,

(ii) $\sum_{t=1}^{\infty} \frac{(E[y(t+1)-y^*(t+1)|F_t])^2}{r(t)} < +\infty$a.s.

Now we claim that $\lim_t r(t) = +\infty$ a.s. Otherwise $r_t = 1 + \sum_{i=1}^{t} \phi^T(k)\phi(k)$ would lead to $\lim_t \phi(t) = 0$ on a set of positive probability. This in turn would imply $\lim_t y_t = 0$ and $\lim_t u_t = 0$, and from the system equation (1) it would then have to follow that $\lim_t C(z)w(t) = 0$ on a set of positive probability, which we will now contradict as follows. First note that $(C(z)w(t))^2$ = a linear combination of terms of the form $w^2(t-i)$ and $w(t-i)w(t-j)$. Let us first examine the first set of square terms. As a consequence of (19f) and Jensen's and Minkowski's inequalities, it follows that $\sup_t E[|w^2(t) - E(w^2(t)|F_{t-1})|^{1+\delta/2}|F_{t-1}]\infty$ a.s. Chow's Theorem [10, Theorem 3.3.1] is therefore applicable, and shows that $\lim_N 1/N \sum_{t=1}^{N} w^2(t) = \sigma^2$ a.s. Now we turn to the cross terms. Since $\sum_{t=1}^{\infty} w^2(t-i) = \infty$ a.s., an appeal to the Local Convergence Theorem for Martingales [11, Lemma 2.3] shows that $\sum_{t=1}^{N} 2w(t-i)w(t) = o(\sum_{t=1}^{N} w^2(t-i))$ a.s. Hence $\lim_N 1/N \sum_{t=1}^{N} w(t-i)w(t) = 0$ a.s. Adding up the contributions we get $\lim_N 1/N \sum_{t=1}^{N}(C(z)w(t))^2 = 1+\sum_{i=1}^{s} c_i^2 > 0$ a.s. This provides the required contradiction.

Since $\lim_t r(t) = +\infty$ a. s., Kronecker's Lemma is applicable and gives

$$\lim_N \frac{1}{r(N)} \sum_{t=1}^{N}(E[y(t+1) - y^*(t+1|F_t])^2 = 0 a.s.$$

Utilizing the strictly minimum phase property of $B(z)$ it follows that $\{r(N)/N\}$ is bounded a.s., which proves (20c) and (20b). The same arguments as in Lemma 7 and Lemma 9 of [6] yield (20a) and (20d).

Case 2: Linear model following problem with $l \leq s$*.* Just as in (21) we still get

$$\begin{aligned} C(z)E[y(t+1) - y^*(t+1)|F_t] \;\; &= \;\; \sum_{i=1}^{p\vee s}(a_i + c_i)y(t-i+1) + \sum_{i=1}^{q} b_i u(t-i+1) \\ &\quad -C(z)y^*(t+1). \end{aligned}$$

Let $\tilde{y}(t) := y_m(t) - y^*(t)$. Then from (9) and (16) we get

$$\begin{aligned} C(z)y^*(t+1) &= C(z)y_m(t+1) - C(z)\tilde{y}(t+1) \\ &= G(z)y_m(t+1) - C(z)\tilde{y}(t+1) \\ &= G(z)y^*(t+1) + [G(z) - C(z)]\tilde{y}(t+1). \end{aligned}$$

Hence

$$\begin{aligned} C(z)E[y(t+1) - y^*(t+1)|F_t] &= \sum_{i=1}^{p\vee s}(a_i + c_i)y(t-i+1) + \sum_{i=1}^{q} b_i u(t-i+1) \\ &\quad -G(z)y^*(t+1) + [C(z) - G(z)]\tilde{y}(t+1) \\ &= \phi^T(t)\theta^\circ + [C(z) - G(z)]\tilde{y}(t+1). \end{aligned}$$

By the strict positive realness property of $[C(z) - \mu + \delta/2]$, it follows that

$$\begin{aligned} S(n) &:= 2\mu \sum_{t=1}^{n} \Big\{ \phi^T(t)\theta^\circ + [C(z) - G(z)]\tilde{y}(t+1) \\ &\quad - \frac{\mu+\delta}{2} E[y(t+1) - y^*(t+1)|F_t] \Big\} \\ &\quad \cdot E[y(t+1) - y^*(t+1)|F_t] \\ &\geq K \text{ a.s. for all } n, \text{ for some} K. \end{aligned}$$

Defining $M(t) := V(t) + S(t-1)/r(t-1)$, we get

$$\begin{aligned} E[M(t+1)|F_t] &\leq M(t) - \frac{\mu\delta}{r(t)}(E[y(t+1) - y^*(t+1)|\ F_t])^2 + \mu^2 \frac{\phi^T(t)\phi(t)}{r^2(t)}\sigma^2 \\ &\quad + \frac{2\mu}{r(t)} E[y(t+1) - y^*(t+1)|F_t][C(z) - G(z)]\tilde{y}(t+1). \end{aligned}$$

Define $\overline{y}(t) := [C(z) - G(z)]\tilde{y}(t+1)$, and note that by (10), $\sum_{t=1}^{\infty} \overline{y}^2(t) < +\infty$. For any $\rho > 0$, we have

$$2E[y(t+1) - y^*(t+1)|F_t]\overline{y}(t) \leq \rho^2(E[y(t+1) - y^*(t+1)|F_t])^2 + \left(\frac{\overline{y}(t)}{\rho}\right)^2.$$

Hence, choose ρ so small that $(\mu\delta - 2\mu\rho^2) > 0$, and note that

$$\begin{aligned} E[M(t+1)|F_t] &\leq M(t) - \frac{(\mu\delta - 2\mu\rho^2)}{r(t)}(E[y(t+1) - y^*(t+1)|F_t])^2 \\ &\quad + \frac{\mu^2\phi^T(t)\phi(t)}{r^2(t)}\sigma^2 + \frac{2\mu\overline{y}^2(t)}{\rho^2 r(t)}. \end{aligned}$$

Now both of the last two terms are summable, and so we can again use the Positive Near Supermartingale Convergence Theorem. The rest of the proof is similar to the previous case.

By (20a) of the above Theorem, we see that usage of the adaptive controller leads to a value of σ^2 for the average of the square of the tracking error. In order to justify our claim at the beginning of this section that the adaptive controller *minimizes* the average of the square of the tracking error we need to show that no other non-anticipative controller, including possibly controllers which utilize knowledge of the parameters (a_i, b_i, c_i), can realize a smaller value than σ^2 for the average squared tracking error on any set of sample paths of positive measure. This is provided in the following Lemma.

Lemma 4. *Consider the ARMAX system (1). Let $F_t := \sigma(w_s$ for $s \leq t$ and y_i, u_i for $i \leq 0)$ be the σ-algebra generated by the past, and let $\{u_t\}$ be any control sequence chosen so that $u_t \in F_t$, i.e., u_t is F_t-measurable for each $t \geq 0$. Then,*

$$\liminf_N \frac{1}{N} \sum_{t=1}^{N} (y(t) - y^*(t))^2 \quad \geq \quad \sigma^2 \qquad \text{a.s.}$$

Proof: Define

$$g(t-1) \quad := \quad \left[\sum_{i=1}^{p} a_i y(t-i) + \sum_{i=1}^{q} b_i u(t-i) + \sum_{i=1}^{s} c_i w(t-i) \right],$$

and note that $g(t-1) \in F_{t-1}$. Rewrite the system equation (1) as $y(t) = g(t-1) + w(t)$ and get

$$y^2(t) \quad = \quad \frac{1}{N} \sum_{t=1}^{N} g^2(t-1) \left[1 + \frac{\sum_{t=1}^{N} 2g(t-1)w(t)}{\sum_{t=1}^{N} g^2(t-1)} \right] + \frac{1}{N} \sum_{t=1}^{N} w^2(t).$$

Appealing to the Local Convergence Theorem for Martingales [12, Lemma 2.3], we know that except on a null set,

$$\begin{aligned} \textstyle\sum_{t=1}^{N} 2g(t-1)w(t) \quad &= \quad o\left(\textstyle\sum_{t=1}^{N} g^2(t-1)\right) \quad && \text{if } \textstyle\sum_{t=1}^{N} g^2(t) = \infty, \\ &< \quad \infty && \text{if } \textstyle\sum_{t=1}^{N} g^2(t) < \infty. \end{aligned}$$

In either case, therefore, it follows that

$$\liminf_N \frac{1}{N} \sum_{t=1}^{N} g^2(t-1) \left[1 + \frac{\sum_{t=1}^{N} 2g(t-1)w(t)}{\sum_{t=1}^{N} g^2(t-1)} \right] \quad \geq \quad 0 \qquad \text{a.s.}$$

Hence,

$$\begin{aligned}\liminf_N \frac{1}{N}\sum_{t=1}^{N} y^2(t) &\geq \lim_N \frac{1}{N}\sum_{t=1}^{N} w^2(t) \\ &= \sigma^2 \qquad \text{a.s.}\end{aligned}$$

The last equality has been proved in the course of the proof of Theorem 3.

6 Self-Tuning and Convergence

In this section we address the self-tuning and convergence properties of the adaptive controllers.

First due to (2,7) we have the same geometrical properties as in [6]. This gives us the following Lemma, see [6].

Lemma 5.

$$\lim_t \|\theta(t)\| \text{ exists and is finite a.s.} \tag{22a}$$

$$\text{For every } n,\ \lim_t \|\theta(t) - \theta(t-n)\| = 0 \text{ a.s.} \tag{22b}$$

$$\|\theta(t+1)\| \geq \|\theta(t)\| \tag{22c}$$

If there is a random scalar ξ and a random subsequence $\{t_k\}$ such that

$$\lim_k \theta(t_k) = \xi\theta^\circ \text{ a.s.} \tag{22d}$$

then

$$\lim_t \theta(t) = \xi\theta^\circ \text{ a.s.}$$

So in order to prove that $\lim_t \theta(t) = \xi\theta^\circ$ it is sufficient to show that there is just *one subsequence* for almost every sample path along which such a limit exists.

Theorem 6.

(i) *Suppose that $\{y^*(t)\}$ in the general tracking problem is strongly sufficiently rich of order $(s+q)$. Then*

$$\lim_t \theta(t) = \xi\theta^\circ \text{ a.s.} \tag{23}$$

for some a.s. finite nonzero scalar random variable ξ.

(ii) *The result (23) holds in the linear model following problem irrespective of the order of strong sufficient richness of $\{y^*(t)\}$ (using the appropriate definition of θ° as in (8) or (17)).*

Proof: We start with (20b) which can be written as

$$\lim_N \frac{1}{N}\sum_{t=1}^{N}\{[1-A(z)]y(t+1)+zB(z)u(t+1)+[C(z)-1]w(t+1)-\{y^*(t+1)\}^2 = 0. \quad (24)$$

Define the time varying polynomials

$$P(t,z) := \sum_{i=1}^{p\vee s}\alpha_i(t)z^{i-1},$$
$$Q(t,z) := \sum_{i=1}^{q}\beta_i(t)z^{i-1},$$
$$R(t,z) := \begin{cases} \sum_{i=0}^{l-1}\gamma_i(t)z^i & \text{in the linear model following problem with } l \le s, \\ \sum_{i=0}^{s}\gamma_i(t)z^i & \text{otherwise.}\end{cases}$$

We shall interpret z as the backward shift operator. Thus, to illustrate the notation,

$$Q(t,z)x(t) := \sum_{i=1}^{q}\beta_i(t)x(t-i+1): \qquad Q(t,z)B(z)x(t) := \sum_{i=1}^{q}\beta_i(t)\sum_{j=1}^{q}b_jx(t-i-j+2)$$
$$B(z)Q(t,z)x(t) := \sum_{j=1}^{q}b_j\sum_{i=1}^{q}\beta_i(t-j+1)x(t-i-j+2).$$

Though $Q(t,z)B(z)x(t) \neq B(z)Q(t,z)x(t)$, it should be noted that if $\{1/N\sum_{t=1}^{N}x^2(t)\}$ is bounded, then it is true that

$$\lim_N \frac{1}{N}\sum_{t=1}^{N}[Q(t,z)B(z)x(t)-B(z)Q(t,z)x(t)]^2 = 0.$$

To verify this, one needs to use the facts that $\lim_t \|\theta(t)-\theta(t-n)\| = 0$ a.s. and $\{\theta(t)\}$ is bounded a.s.

Multiplying inside the summation in (24) by $Q(t,z)$, we have

$$\lim_N \frac{1}{N}\sum_{t=1}^{N}\{Q(t,z)[1-A(z)]y(t+1)+Q(t,z)zB(z)u(t+1)$$
$$+Q(t,z)[C(z)-1]w(t+1)-Q(t,z)y^*(t+1)\}^2 = 0 \text{ a.s.}$$

Since

$$\left\{\frac{1}{N}\sum_{t=1}^{N}y^2(t)\right\},\left\{\frac{1}{N}\sum_{t=1}^{N}u^2(t)\right\},\left\{\frac{1}{N}\sum_{t=1}^{N}w^2(t)\right\},\left\{\frac{1}{N}\sum_{t=1}^{N}y*^2(t)\right\}$$

are all bounded, we can interchange the polynomials above to get

$$\lim_N \frac{1}{N}\sum_{t=1}^{N}\{z^{-1}[1-A(z)]Q(t,z)y(t)+B(z)Q(t,z)u(t) \qquad (25)$$

$$+z^{-1}[C(z)-1]Q(t,z)w(t)-Q(t,z)\{y^*(t+1)\}^2 \;=\; 0 \text{ a.s.}$$

Now note that the control laws (5) and (14) can be written as

$$Q(t,z)u(t) \;=\; -P(t,z)y(t)+R(t,z)y^*(t+1). \qquad (26)$$

Substituting (26) in (25) gives

$$\lim_N \frac{1}{N}\sum_{t=1}^{N}\{\{z^{-1}[1-A(z)]Q(t,z)-B(z)P(t,z)\}y(t)$$

$$+z^{-1}[C(z)-1]Q(t,z)w(t)$$

$$+B(z)R(t,z)-Q(t,z)\}y^*(t+1)\}^2 \;=\; 0\text{a.s.}$$

Now $y(t)=w(t)+y^*(t)+E[y(t)-y^*(t)|F_{t-1}]$, and so substituting for $y(t)$ gives

$$\lim_N \frac{1}{N}\sum_{t=1}^{N}\{\{z^{-1}[C(z)-A(z)]Q(t,z)-B(z)P(t,z)\}w(t)$$

$$+\{B(z)R(t,z)-zB(z)P(t,z)-A(z)Q(t,z)\}y^*(t+1)$$

$$+\{z^{-1}[1-A(z)]Q(t,z)-B(z)P(t,z)E[y(t)\}-y^*(t)|F_t]\}^2 \;=\; 0\text{a.s.}$$

Due to (20b) and the fact that $\{\theta(t)\}$ is bounded, we can drop the last term above and write

$$\lim_N \frac{1}{N}\sum_{t=1}^{N}\{\{z^{-1}[C(z)-A(z)]Q(t,z)-B(z)P(t,z)\}w(t)$$

$$+\{B(z)R(t,z)-zB(z)P(t,z)-A(z)Q(t,z)\}y^*(t+1)\}^2 \;=\; 0\text{a.s.}$$

Since $\lim_t \|\theta(t)-\theta(t-1)\|=0$ a.s., and since $\{y^*(t+1)\}$ is bounded, we can replace $R(t,z)$, $P(t,z)$ and $Q(t,z)$ above by $R(t-n,z)$, $P(t-n,z)$ and $Q(t-n,z)$, respectively, for any n. Thus

$$\lim_N \frac{1}{N}\sum_{t=1}^{N}\{\{z^{-1}[C(z)-A(z)]Q(t-n,z)-B(z)P(t-n,z)\}w(t)$$

$$+\{B(z)R(t-n,z)-zB(z)P(t-n,z)-A(z)Q(t-n,z)\}y^*(t+1)\}^2 \;=\; 0\text{a.s}$$

Choose n larger than $(p+q+s)$, and then we can apply Lemma 11 of [6] to deduce that

$$\lim_N \frac{1}{N}\sum_{t=1}^{N}\{z^{-1}[C(z)-A(z)]Q(t-n,z)-B(z)P(t-n,z)\}^2 \;=\; 0 \text{ a.s.} \tag{27}$$

by which we mean that the average of the square of each coefficient of the polynomial in z is 0; and also

$$\begin{aligned}\lim_N \frac{1}{N}\sum_{t=1}^{N}\{\{B(z)R(t-n,z)-zB(z)P(t-n,z)\\ -A(z)Q(t-n,z)\}y^*(t+1)\}^2 \;&=\; 0\text{a.s.}\end{aligned} \tag{28}$$

Furthermore since $\{y^*(t)\}$ is bounded, (27) also implies that

$$\lim_N \frac{1}{N}\{\{[C(z)-A(z)]Q(t-n,z)-zB(z)P(t-n,z)\}y^*(t+1)\}^2 \;=\; 0\text{a.s.} \tag{29}$$

Subtracting (28) appropriately from (29), we get

$$\lim_N \frac{1}{N}\sum_{t=1}^{N}\{[C(z)Q(t-n,z)-B(z)R(t-n,z)]y^*(t+1)\}^2 \;=\; 0 \text{ a.s.} \tag{30}$$

Changing $t-n$ back to t in (27) and (30), we arrive at

$$\lim_N \frac{1}{N}\sum_{t=1}^{N}\{z^{-1}[C(z)-A(z)]Q(t,z)-B(z)P(t,z)\}^2 \;=\; 0 \text{ a.s.} \tag{31}$$

$$\lim_N \frac{1}{N}\sum_{t=1}^{N}\{[C(z)Q(t,z)-B(z)R(t,z)]y^*(t+1)\}^2 \;=\; 0 \text{ a.s.} \tag{32}$$

Now let us treat the cases separately.

Case 1: Strong sufficient richness of order greater than or equal to $(q+s)$. This case includes the general tracking problem as well as the linear model following problem with the order of sufficient richness as shown. Since $\{y^*(t)\}$ is strongly sufficiently rich of order greater than or equal to $(q+s)$, there exist n and $\epsilon > 0$ such that for all large t,

$$\begin{aligned}\tfrac{1}{n}\textstyle\sum_{k=t+1}^{t+n}(y^*(k+1),\ldots,y^*(k-q-s+2))(y^*(k+1),\ldots,\\ y^*(k-q-s+2))^T \geq \epsilon I_{s+q}.\end{aligned} \tag{33}$$

Define

$s_0(t) + s_1(t)z + \cdots + s_{q+s-1}(t)z^{q+s-1} := S(t, z) := C(z)Q(t, z) - B(z)R(t, z).$

Then (32) can also be written as

$$\lim_m \frac{1}{m}\sum_{j=1}^{m}\left\{\frac{1}{n}\sum_{k=jn+1}^{jn+n}[S(k, z)y^*(k+1)]^2\right\} \quad = \quad 0 \text{ a.s.}$$

Since $\lim_t \|\theta(t) - \theta(t-1)\| = 0$, we can replace $S(k, z)$ by $S(jn, z)$ to get

$$\lim_m \frac{1}{m}\sum_{j=1}^{m}\left\{\frac{1}{n}\sum_{k=jn+1}^{jn+n}[S(jn, z)y^*(k+1)]^2\right\} \quad = \quad 0 \text{ a.s.} \qquad (34)$$

Define $\|S(t, z)\|^2 := \sum_{i=0}^{q+s-1} s_i^2(t)$ and (33) implies that

$$\frac{1}{n}\sum_{k=jn+1}^{jn+n}[S(jn, z)y^*(k+1)]^2 \quad \geq \quad \epsilon\|S(jn, z)\|^2 \text{ for all large} j.$$

From (34) it follows that

$$\lim_m \frac{1}{m}\sum_{j=1}^{m}\|S(jn, z)\|^2 \quad = \quad 0 \text{ a.s.} \qquad (35)$$

Again, since $\lim_t \|\theta(t) - \theta(t-1)\| = 0$ a.s., (35) implies that

$$\lim_N \frac{1}{N}\sum_{t=1}^{N}\|S(t, z)\|^2 = 0 \text{ a.s.} \qquad (36)$$

Adding (31) and (36) gives

$$\lim_N \frac{1}{N}\sum_{t=1}^{N}\{z^{-1}[C(z) - A(z)]Q(t, z) - B(z)P(t, z)\}^2$$
$$+\{C(z)Q(t, z) - B(z)R(t, z)\}^2 \quad = \quad 0 \text{ a.s.}$$

Hence there is a common subsequence $\{t_k\}$ such that

$$\lim_k\{z^{-1}[C(z) - A(z)]Q(t_k, z) - B(z)P(t_k, z)\} \quad = \quad 0 \text{ a.s.} \qquad (37)$$

and

$$\lim_k \{C(z)Q(t_k,z) - B(z)R(t_k,z)\} \quad = \quad 0 \text{ a.s.} \tag{38}$$

Since $\{\theta(t)\}$ is bounded, we can also assume without loss of generality that

$$\lim_k Q(t_k,z) =: Q(z); \quad \lim_k P(t_k,z) =: P(z); \quad \lim_k R(t_k,z) =: R(z) \text{ a.s.} \tag{39}$$

exist. Hence (37) and (38) imply

$$z^{-1}[C(z) - A(z)]Q(z) - B(z)P(z) \quad = \quad 0 \text{ a.s.} \tag{40}$$

$$C(z)Q(z) - B(z)R(z) \quad = \quad 0 \text{ a.s.} \tag{41}$$

However, $Q(z)$ and $P(z)$ are polynomials of degrees less than or equal to $(q-1)$ and $(p \vee s - 1)$, respectively. Hence (40) and our assumption (19d) imply that

$$Q(z) = \xi B(z) \quad \text{and} \quad P(z) = \xi z^{-1}[C(z) - A(z)] \tag{42}$$

for some random scalar ξ. Then (41) also shows the $R(z) = \xi C(z)$. Moreover ξ cannot be 0, since otherwise $\lim_k \theta(t_k) = 0$, which is ruled out by (22c) and (19g). From (22d) we obtain the desired result.

Case 2: Linear model following problem with $l(q+s)$. Since $\lim_t(y_m(t) - y^*(t)) = 0$, we can replace $y*(t+1)$ by $y_m(t+1)$ in (32). If $s+1 \leq lq+s$, we shall henceforth define $G(z) := C(z)$, while if $l \leq s$, $G(z)$ is defined as previously by (15) and (16). In the latter case also, from (9) and (17) we have $C(z)y_m(t+1) = G(z)y_m(t+1)$. Hence in any case,

$$\lim_N \frac{1}{N} \sum_{t=1}^{N} [G(z)Q(t,z) - B(z)R(t,z)]\{y_m(t+1)\}^2 = 0 \text{ a.s.} \tag{43}$$

Applying Lemma 2 to (43) and (31) we obtain that there is a subsequence $\{t_k\}$ such that (37) holds and also

$$\lim_k [G(z)Q(t_k,z) - B(z)R(t_k,z)] = K(z)H(z) \qquad \text{a.s.}$$

Without loss of generality we can also suppose that the limits in (39) exist. Hence

$$G(z)Q(z) - B(z)R(z) = K(z)H(z) \qquad \text{a.s.} \tag{44}$$

Also through (31), (40) gives (42). Substituting (42) in (44) yields

$$B(z)[\xi G(z) - R(z)] = K(z)H(z) \qquad \text{a.s.}$$

Now note that by (11b) all the roots of $H(z)$ are exactly on the unit circle, while all the roots of $B(z)$ are strictly outside the unit circle by (19a). Hence

$$\xi G(z) - R(z) = J(z)H(z) \qquad \text{a.s.}$$

for some polynomial $J(z)$. However $[\xi G(z) - R(z)]$ is a polynomial of degree less than or equal to $l-1$, while $H(z)$ is a polynomial of degree exactly l. Hence

$$\xi G(z) - R(z) = 0 \qquad \text{a.s.} \tag{45}$$

(42) and (45) now yield the theorem.

It is of interest to note that Caines and Lafortune [8] have suggested an adaptive controller which tracks $y^*(t)$ perturbed by white noise. Such a perturbed reference trajectory is strongly sufficiently rich of arbitrary large order (effectively ∞).

Having proved convergence of the parameters to $\xi\theta^\circ$ under the conditions of Theorem 6, we now have the following results.

Theorem 7.

(i) *In the general tracking problem suppose $\{y^*(t)\}$ is strongly sufficiently rich of order greater than or equal to $(q+s)$. Then*

$$\lim_t \frac{1}{\gamma_0(t)}(\alpha_1(t) - \gamma_1(t), \ldots, \alpha_p(t) - \gamma_p(t), \beta_1(t), \ldots, \beta_q(t), \gamma_1(t), \ldots, \gamma_s(t)) \tag{46}$$
$$= (a_1, \ldots, a_p, b_1, \ldots, b_q, c_1, \ldots, c_s) \text{ a.s.}(\textit{with } \gamma_i(t) := 0 \textit{ for } i > s).$$

Thus the parameter estimates are strongly consistent. Also

$$\lim_t \frac{1}{\beta_1(t)}(\alpha_1(t), \ldots, \alpha_{p\vee s}(t), \beta_2(t), \ldots, \beta_q(t), \gamma_0(t), \ldots, \gamma_s(t)) \tag{47}$$
$$= \frac{1}{b_1}(a_1 + c_1, \ldots, a_{p\vee s} + c_{p\vee s}, b_2, \ldots, b_q, 1, c_1, \ldots, c_s) \text{ a.s.}$$

setting $a_i := 0$ for $i > p$ and $c_i := 0$ for $i > s$. Hence the adaptive control law (5) self-tunes to the optimal control law a.s.

(ii) *In the linear model following problem with $l > s$ the results (46) and (47) continue to hold.*

(iii) *In the linear model following problem with $l \leq s$ we have,*

$$\lim_t \frac{1}{\beta_1(t)}(\alpha_1(t), \ldots, \alpha_{p \vee s}(t), \beta_2(t), \ldots, \beta_q(t), \gamma_0(t), \ldots, \gamma_{l-1}(t))$$
$$= \frac{1}{b_1}(a_1 + c_1, \ldots, a_{p \vee s} + c_{p \vee s}, b_2, \ldots, b_q, g_0, \ldots, g_{l-1})$$

setting $a_i := 0$ for $i > p$ and $c_i := 0$ for $i > s$. Here $\{g_{0,} \ldots, g_{l-1}\}$ are defined by (15), (16). Hence the adaptive control law self-tunes to the optimal control law a.s.

7 Concluding Remarks

We have proved the convergence of the parameter estimates and the self-tuning property for the *adaptive tracking* problem, justifying the name of *self-tuning trackers.*

For the *general tracking problem*, the convergence depends on whether the reference trajectory is sufficiently rich of appropriate order, as shown in Theorem 7. In the important case of reference trajectories which are not so rich, we have examined the *linear modeling problem*, and shown how one can adjust the *dimension* of the parameter estimator to the order of sufficient richness so as to obtain a self-tuning tracker. It is worth noting that the adaptive controller need not be provided with precise information such as amplitude, frequency or phases of the sinusoids in the reference trajectory. It is enough to know only the number of such components.

An important application, which is a special case of these results, is the problem of maintaining the output at a constant level, i.e., the *set-point problem.* The constant trajectory is sufficiently rich of only order 1, and only one parameter need be estimated to compensate for the colored noise and reject it optimally.

Among the outstanding problems still left unresolved are the following:

(i) Does the least squares based parameter estimation algorithm also possess the above properties? This is of vital interest because the rate of convergence of least squares based algorithms has been observed to be superior to the type of parameter estimation algorithm considered here.

(ii) What robustness properties do these types of self-tuning adaptive control laws possess?

Acknowledgement. The authors would like to thank P. Kokotovic for several stimulating discussions.

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