OPTIMALITY OF ZERO-INVENTORY POLICIES FOR UNRELIABLE MANUFACTURING SYSTEMS

T. Bielecki*and P. R. Kumar†

October 15, 1999

Abstract

We show that there are ranges of parameter values describing an unreliable manufacturing system for which *zero-inventory policies* are exactly optimal even when there is uncertainty in manufacturing capacity. This result may be initially surprising since it runs counter to the argument that inventories are buffers against uncertainty and that therefore one must strive to maintain a strictly positive inventory as long as there is any uncertainty. However, there is a deeper reason why this argument does not hold, and why a zero-inventory policy can be optimal even in the presence of uncertainty. This provable optimality reinforces the case for zero-inventory policies, which is currently made on the separate grounds that it enforces a healthy discipline on the entire manufacturing process.

In recent years, the goals of "zero-inventory" and "stockless production" have attracted much attention (Hall 1983). Such policies enforce a strict discipline on the entire manufacturing process that has several beneficial consequences for the overall production system.

In this paper, we examine the question of whether there are any conditions under which a zero-inventory policy is actually *optimal*. In this connection it should be noted that it is sometimes argued that a zero-inventory policy only passes the costs from, say, the assembly line to the parts manufacturer, and that, therefore, any advantages accruing from enforcing such a policy are only side benefits resulting from the greater discipline governing the system, and not directly from the policy itself. After all, the reasoning goes, positive inventories are used as a buffer against uncertainties in supply or demand, and so as long as there is any uncertainty whatsoever in the system, one should strive to maintain

*(Please address all correspondence to the second author below). Main College of Planning and Statistics, Warsaw, Poland.

†Department of Electrical and Computer Engineering and Coordinated Science Laboratory, University of Illinois, 1101 W. Springfield Avenue, Urbana, Illinois 61801/USA.

a strictly positive buffer. Thus, zero inventory levels can only be optimal when there is no uncertainty at all, and that is never possible !

In contrast to this conclusion, we show in this paper that there are conditions under which a zero-inventory policy is actually *provably optimal*, even when there is uncertainty, and that furthermore such optimality of the zero-inventory policy results whenever the system is designed to be *sufficiently*, but not necessarily perfectly, efficient.

The model with respect to which we exhibit these conclusions is that of a simple failure prone manufacturing system producing a commodity. The time between failures is random and modeled as an exponentially distributed random variable with mean $1/q_1$, while the repair time is exponentially distributed with mean $1/q_0$. When functioning, the system can produce at any rate up to a maximum of r units/time; while broken down it cannot produce at all. At all times the commodity is being depleted at a demand rate of d units/time. The total inventory can be negative, which just corresponds to a backlog. Positive inventories are assessed a cost at a rate of c^+ dollars per unit commodity per unit time, while negative inventories are assessed a similar cost of c^-. We seek the optimal production policy for this system which minimizes the long run average expected cost incurred per unit time,

$$\lim_{T\to\infty} \frac{1}{T} E \int_0^T [c^+ x^+(t) + c^- x^-(t)] dt \tag{1}$$

where $x(t) =$ inventory level at time t (see below), $x^+ = \max(0, x)$ is its positive part, and $x^- = \max(0, -x)$ its negative part. Let $u(t)$ be the production rate at time t, then clearly,

$$x(t) \quad = \quad \int_0^t (u(s) - d) ds$$

is the inventory at time t, and let $I(t) = 1$ or 0, respectively, depending on whether the system is functioning or not. Clearly, $u(t) = 0$ whenever $I(t) = 0$, and so we only need to determine the optimal production rate when the manufacturing system is up, i.e., in state $I(t) = 1$.

We show that there is a number z^*, called the *optimal inventory level*, toward which the production should be aimed, i.e.,

$$\begin{aligned} u(t) &= 0 \text{ if } x(t) z^*, I(t) = 1 & (2)\\ &= d i f x(t) = z^*, I(t) = 1 & (3)\\ &= r i f x(t) z^*, I(t) = 1. & (4) \end{aligned}$$

Thus, if current inventory $x(t)$ exceeds z^*, the system should produce nothing; if $x(t)$ is less than the optimal level z^*, the system should produce at the maximum rate r; if the inventory level exactly equals z^*, then the system should produce exactly enough to meet demand and thereby keep the inventory level at z^*.

The quantity z^*, the optimal inventory level, is therefore key to the whole problem, and it is explicitly given by:

$$z^* = 0 \text{ if } \frac{rq_1(c^+ + c^-)}{c^+(r-d)(q_0+q_1)} \leq 1 \text{ and } \frac{r-d}{q_1}\frac{d}{q_0} \tag{5}$$

$$= +\infty \text{ if } \frac{r-d}{q_1}\frac{d}{q_0} \tag{6}$$

$$= \frac{1}{((q_0/dd) - (q_1/(r-d)))} \log\left[\frac{rq_1(c^+ + c^-)}{c^+(r-d)(q_0+q_1)}\right] \text{ otherwise.} \tag{7}$$

Note that $1/q_1$ is the mean up-time and so $r - d/q_1$ is the maximum total production in a mean up-time. Similarly $1/q_0$ is the mean down-time and so d/q_0 is the total depletion in a mean down-time. So if $r - d/q_1 < d/q_0$, then the system does not have the capacity to meet demand even if it produces at full capacity when up. Thus, $z^* = +\infty$ under these conditions.

So consider only the case $r - d/q_1 > d/q_0$. Then we have the interesting situation that the optimal inventory level z^* is *exactly* 0 whenever

$$\frac{rq_1(c^+ + c^-)}{c^+(r-d)(q_0+q_1)} \quad \leq 1.$$

This happens whenever q_1 is reduced enough or q_0 increased enough, i.e., whenever the system is made efficient. Note that it is *not* necessary to have $q_1 = 0$ or for $q_0 = +\infty$ in order for a zero-inventory policy to be optimal. This is somewhat surprising because positive buffers are usually maintained as a hedge against future uncertainty (about manufacturing capacity, in our case), and so one feels that a strictly positive inventory should be maintained as long as there is *any* uncertainty, i.e., as long as $q_0 \neq +\infty$ or $q_1 \neq 0$, in the system. However, there is a deeper phenomenon at work, as we explain in Section I.

The above manufacturing system is a special case of the multi-commodity, multi-machine, failure-prone flexible manufacturing system considered by Kimemia and Gershwin (1983), which in fact has been a key motivation for us. In Akella and Kumar (1986), the authors have considered a similar problem but with a discounted cost criterion,

$$E\int_0^\infty e^{-\gamma t}[c^+x^+(t) + c^-x^-(t)]dt \tag{8}$$

where $\gamma > 0$ is a discount factor. However, the average cost case (1) considered here admits a very nice interpretation for the fundamental reason behind the optimality of a zero-inventory policy; see Section 1. The precise proof of optimality, though, which is necessary, and which we provide in Sections 2 and 3, is more intricate.

We hope that the explicit expression for the optimal inventory level z^*, given by (5-7), will find use as a guideline. For example, the condition

$$\frac{c^+(r-d)(q_0+q_1)}{rq_1(c^+ + c^-)} \quad \geq \quad 1$$

shows what range of values of c^+, c^-, r, d, q_0 and q_1 one should strive for in order to make the implementation of a zero-inventory policy optimal. If one cannot achieve such a range of parameter values, then the formula

$$z^* \quad = \quad \left(1 \Big/ \left(\frac{q_0}{d} - \frac{q_1}{r-d}\right)\right) \log\left[\frac{rq_1(c^+ + c^-)}{c^+(r-d)(q_0+q_1)}\right]$$

tells us the level of the positive inventory that one should aim for, and the manner in which this level varies with the parameters.

1 WHY A ZERO-INVENTORY POLICY CAN BE OPTIMAL

In this section we will provide an intuitive explanation for why a zero-inventory policy is optimal, contrary to the expectation that positive inventories should be maintained whenever there is any supply uncertainty.

Consider policies of the form (2-4), i.e., policies for which there is a z such that

$$u(t) = r \text{ if } x(t) < z, \qquad I(t) = 1 \tag{9}$$
$$= d \text{ if } x(t) = z, \qquad I(t) = 1 \tag{10}$$
$$= 0 \text{ if } x(t) z, \qquad I(t) = 1 \tag{11}$$
$$= 0 \text{ if } I(t) = 0. \tag{12}$$

The only quantity to be chosen is z, and we shall optimize over its possible values. (This is a considerable restriction of the class of all policies, and in fact the proof of Sections II and III is needed precisely because one does not know a priori that other types of policies can be ruled out.)

We will assume that the combined process $(x(t), I(t))$ has a steady state probability distribution,

$$P^z(A,i) \quad := \quad \lim_{t\to\infty} Prob(x(t) \in A, I(t) = i)$$

when the policy (9-12) is used. Define

$$Q^z(A) \quad := \quad P^z(A,0) + P^z(A,1).$$

The average inventory cost (1) corresponding to this policy, which we denote by $J(z)$, can then be computed as the expected cost with respect to the steady state distribution, i.e., it can be written as,

$$J(z) \quad = \quad \int_{-\infty}^{0} c^{-}|x|Q^{z}(dx) + \int_{0}^{\infty} c^{+}xQ^{z}(dx). \tag{13}$$

We will shortly minimize $J(z)$ over z to obtain z^{*}.

We claim first that

$$P^{z}((z,\infty),i) = 0 \text{ for } i = 0,1 \tag{14a}$$

$$P^{z}(\{z\},1) =: \alpha > 0. \tag{14b}$$

Note from (11 and 12) that if the inventory level $x^{z}(0)$ starts with a value larger than z, then it is depleted at rate d until it hits z. Thereafter (10 and 12) ensure that it never again rises above z; see Figure 1. Hence, $x^{z}(t) \leq z$ for all t large

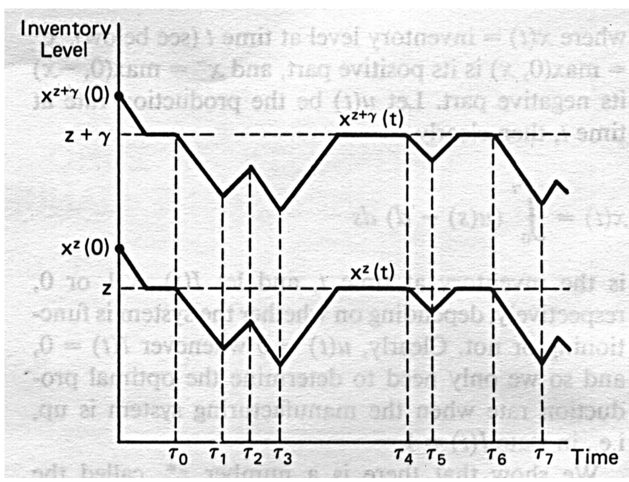

Figure 1: Comparison of inventory behavior under policies with z and $z+\gamma$, when $x^{z+\gamma}(0) = x^{z}(0) + \gamma$.

enough, and so (14a) follows. To see (14b) which asserts that there is a strictly positive *probability mass* at z, note that whenever $x(t)$ hits z with $I(t) = 1$, it stays at z until I switches to 0. So $x(t)$ does spend a positive fraction of time at exactly the level z, which implies that the point z has positive probability mass, giving (14b).

Lastly, let us make the reasonable assumption that except for the mass $P^{z}(\{z\},1)$, the distribution $P^{z}(dx,i)$ has a probability *density* function $p_{i}^{z}(x)$. Hence $Q^{z}(dx)$ also has a probability density function $q^{z}(x)$, except for the mass $Q^{z}(\{z\})$ at z.

The property (14b) is key to the optimality of a zero-inventory policy. By choosing $z = 0$ one can ensure that with probability the system has zero inventory, and therefore *with probability one is paying no inventory cost.* Of course, we must balance against this the possibly extra cost incurred by choosing $z = 0$. If is *large enough*, then the advantages will outweigh the disadvantages and so a zero inventory policy with $z = 0$ is optimal. This is at heart the main reason for optimality of zero inventory policy.

To decide when is large enough to make it advantageous to choose $z = 0$, we need to compare costs with $z = 0$ and $z \neq 0$. For doing this it is helpful to note that translating z merely translates the probability distribution P^z, that is

$$P^z(A, i) \quad = \quad P^{z+\gamma}(A + \gamma, i) \text{ for all} \gamma, \tag{15}$$

where by the notation $A + \gamma$ we mean the set A translated by γ, or

$$A + \gamma \quad := \quad \{x : (x - \gamma) \in A\}.$$

To see (15), suppose that $\{\tau_i\}$ are the switching times of $I(t)$, i.e.,

$$\begin{aligned} I(t) \quad &= \quad 0 \text{ for } \tau_{2n} \leq t \tau_{2n+1} \\ &= \quad 1 \text{ for } \tau_{2n+1} \leq t \tau_{2n+2}. \end{aligned}$$

If $\{x^z(t), I(t)\}$ and $\{x^{z+\gamma}(t), I(t)\}$ are the processes corresponding to the choices of z and $z + \gamma$, with *initial choice* of

$$x^{z+\gamma}(0) \quad = \quad x^z(0) + \gamma,$$

then from (9-12) we will have

$$x^{z+\gamma}(t) \quad = \quad x^z(t) + \gamma \text{ for all } t \geq 0.$$

Again, see Figure 1. Since the paths are translates of each other, so are the stationary probability distributions.

We can now optimize $J(z)$ quite readily. From (15) it follows that moving from a choice of 0 to a *negative* value of z merely shifts the distribution P^0 leftwards by z units (thereby resulting in P^z). From (14a) it follows that

$$J(z) - J(0) \quad = \quad c^-|z| \text{ for } z \leq 0.$$

Hence, the optimal z will *never* be negative.

Now compare the advantage in going from a choice of 0 to a *positive* value of z. There are three components; translating the probability distribution on $(-\infty, -z)$ to $(-\infty, 0)$ yields an advantage in cost of $c^- z Q^0(-\infty, -z)$, the movement of the probability mass from 0 to z costs $\alpha c^+ z$, and translating the distribution over $(-z, 0)$ to $(0, z)$ yields a difference of

$$\int_{-z}^{0} [c^-|x| - c^+(x + z)] Q^0(dx).$$

So,

$$
\begin{aligned}
J(z) - J(0) &= [\alpha c^+ z - c^- z Q^0((-\infty, -z))] \\
&\quad + \int_{-z}^{0} (c^+(x+z) + c^- x) Q^0(dx)], \qquad (16) \\
&\quad \text{for} z 0.
\end{aligned}
$$

The derivative at 0 is

$$
\frac{d}{dz}(J(z) - J(0))]_{z=0} = \alpha c^+ - c^- Q^0((-\infty, 0)).
$$

For a zero-inventory policy to be optimal, it is necessary that this be positive, which happens when

$$
\alpha c^+ \geq c^- Q^0((-\infty, 0)), \qquad (17)
$$

i.e., the cost of the probability mass at 0 is large enough. Moreover, for $z > 0$, $(d/dz)[J(z) - J(0)]$ can be computed to be,

$$
\begin{aligned}
&\frac{d}{dz}[J(z) - J(0)]_{z 0} \\
&= \alpha c^+ - (1-\alpha) c^- + (c^+ + c^-) Q^0((-z, 0)),
\end{aligned}
$$

which is also positive when (17) holds. Hence (17) is both necessary and sufficient for $z = 0$ to yield a minimum of $J(z)$.

The basic reason for the optimality of an exactly zero inventory policy should now be intuitively clear.

Using $Q^0((-\infty, 0)) = 1 - \alpha$, we can rewrite (17) as

$$
\alpha(c^+ + c^-) \geq c^-. \qquad (18)
$$

Next, we need to compute α. This is done by determining $p_i^z(\cdot)$ and α in steady state. (Recall that $p_i^z(\cdot)$ is the density function of $P^z(dx, i)$ in $(-\infty, z)$). Let $p_i^{z,t}(\cdot)$ and α^t denote the corresponding quantities at time t. Considering the probabilities at time $(t+h)$, for small $h > 0$, we have:

$$
\begin{aligned}
\int_A p_0^{z,t+h}(x)dx &= (1 - q_0 h) \int_{A+hd} p_0^{z,t}(y)dy \\
&\quad + q_1 h \int_{A-(r-d)h} p_1^{z,t}(y)dy + o(h) \\
\int_A p_1^{z,t+h}(x)dx &= q_0 h \int_{A+hd} p_0^{z,t}(y)dy \\
&\quad + (1 - q_1 h) \int_{A-(r-d)h} p_1^{z,t}(y)dy + o(h), \\
\text{for } z \notin A & \\
\alpha^{t+h} &= (1 - q_1 h)\alpha^t + \int_{z-(r-d)h}^{z} p_1^{z,t}(y)dy + o(h).
\end{aligned}
$$

In equilibrium we should have $p_i^{z,t+h} \equiv p_i^{z,t}$ and $\alpha^{t+h} = \alpha^t$. Taking the limit as $h \to 0$ in equilibrium, and proceeding formally through,

$$\begin{aligned}
\int_{A+hd} p_0^z(y)dy &= \int_A p_0^z(y+hd)dy \\
&= \int_A \left(p_0^z(y) + \frac{dp_0^z(y)}{dy} hd \right) dy + o(h),
\end{aligned}$$

we get the differential equations

$$\begin{aligned}
(-d)\frac{dp_0^z(x)}{dx} &= -q_0 p_0^z(x) + q_1 p_1^z(x) && (19)\\
(r-d)\frac{dp_1^z(x)}{dx} &= q_0 p_0^z(x) - q_1 p_1^z(x) && (20)\\
\alpha &= \frac{(r-d)}{q_1} p_1^z(z). && (21)
\end{aligned}$$

Using the auxiliary condition

$$\int_{-\infty}^{z} p_0^z(x)dx = \frac{q_1}{q_0+q_1} \qquad (22)$$

one can solve these differential equations to get,

$$\begin{aligned}
p_0^z(x) &= \frac{\lambda q_1}{q_0+q_1} e^{\lambda(x-z)} \text{ for } x \le z && (23)\\
p_1^z(x) &= \frac{\lambda q_1 d}{(q_0+q_1)(r-d)} e^{\lambda(x-z)} for x \le z && (24)\\
\alpha &= \frac{\lambda d}{q_0+q_1}, && (25)
\end{aligned}$$

where the constant λ is defined as,

$$\lambda := \frac{q_0}{d} - \frac{q_1}{r-d}. \qquad (26)$$

Recall that we have assumed,

$$\lambda > 0, \qquad (27)$$

(see 5 and 6).

By utilizing these computations, it is easy to verify that (18) is equivalent to the condition

$$\frac{c^+(r-d)(q_0+q_1)}{rq_1(c^+ + c^-)} \ge 1$$

in (5). When this condition is not satisfied, straightforward evaluation of the minimum of (16) using (25)-(27) yields the minimizing value of z^* presented in (7).

We summarize below the computed values for the optimal cost $J(z^*)$:

$$\begin{aligned} J(z^*) &= \frac{c^- q_1 r d}{(q_0+q_1)(q_0 r - q_0 d - q_1 d)} \text{ when } z^* = 0, \\ &= \frac{c^+ d}{q_0+q_1} + \left(c^+ \middle/ \frac{c^+}{\frac{q_0}{d} - \frac{q_1}{r-d}} \right) \\ &\quad \times \log \left[\frac{r q_1 (c^+ + c^-)}{c^+ (r-d)(q_0+q_1)} \right] \text{ when } z^* > 0. \end{aligned}$$

2 The Verification Theorem

Let $u = \pi(x, i)$ be a policy. By this we mean that the inventory evolves as

$$\frac{dx(t)}{dt} = \pi(x(t)), I(t)) - d \tag{28}$$

where $I(t)$ is the Markov chain denoting whether the system is up or down.

Some conditions are needed on the function π for there to exist a solution of the differential equation when $\pi(\cdot, i)$ is discontinuous (as the optimal policy is). We will assume the following

$$\pi(x,0) = 0 \quad \text{and} \quad 0 \le \pi(x,1) \le r, \tag{29a}$$

$\pi(\cdot, 1)$ is measurable, and for every $(\tau, \xi) \in R^2$ with $\tau \ge 0$, there exists a unique function $y_\pi(t; \tau, \xi)$ which is absolutely continuous in t, continuous in (t, τ, ξ), and which satisfies

$$y_\pi(t;\tau,\xi) = \xi + \int_\tau^t (\pi(y_\pi(s;\tau,\xi),1) - d)ds \quad \text{for } t \ge \tau, \tag{29b}$$

$$\lim_{t \downarrow \tau} \frac{1}{(t-\tau)} \int_\tau^t \pi(y_\pi(s;\tau,\xi),1)ds = \pi(\xi,1), \tag{29c}$$

$$\lim_{T\to\infty} \frac{1}{T} E_\pi[x^2(T)] = 0. \tag{29d}$$

We shall say that a policy is *admissible* if it satisfies (29a, 29b, 29c and *stable* if it satisfies (29d).

The reason for imposing (29a) is clear. The condition (29b) guarantees a solution of the differential equation (28) while still allowing discontinuous functions π; see Akella and Kumar where sufficient conditions are given. The condition (29c) rules out some pathological functions and allows the definition of an infinitesimal operator. Finally, (29d) denotes a stability property that we want π to possess. Here and in what follows, E_π denotes the expected value when a policy π is used.

We now have the following theorem which gives sufficient conditions of the Dynamic Programming type for a policy to be optimal with respect to the average cost criterion. In what follows we use the notation, $c(x) := c^+x^+ + c^-x^-$.

Theorem. *Suppose π^* is an admissible stable policy, and there exist continuously differentiable functions $W(x,i)$ and a constant J^* such that*

$$[\pi^*(x,i)-d]\frac{dW(x,i)}{dx}$$
$$-q_i[W(x,i)-W(x,1-i)]+c(x)-J^*=0$$
$$\text{for } i=0,1 \text{ and all } x \tag{30a}$$
$$[\pi^*(x,1)-d]\frac{dW(x,1)}{dx} = \min_{0\le u\le r}[u-d]\frac{dW(x,1)}{dx} \tag{30b}$$
$$|W(x,i)| \le k_1x^2+k_2 \text{ for some } k_1,k_2. \tag{30c}$$

Then

$$\lim_{T\to\infty}\frac{1}{T}E_{\pi^*}\int_0^T c(x(t))dt = J^* \le \lim\infty_{T\to\infty}\frac{1}{T}E_\pi\int_0^T c(x(t))dt$$

for all admissible stable π and all initial conditions $x(0)$.

Proof: The proof uses the (main) Dynkin formula, see Karlin and Taylor (1981). To compute the infinitesimal operator, let $f(x,i)$ be continuously differentiable in x and vanishing outside a compact set. Then for an admissible π we have

$$\begin{aligned}
&E_\pi[f(x(h),I(h))|x(0)=x,I(0)=i] \\
&= Prob(1 \text{ jump of } I \text{ in } (0,h))E_\pi[f(x(h),I(h))| \\
&\quad x(0)=x,I(0)=i,1 \text{ jump of } I \text{ in } (0,h)] \\
&\quad +Prob(0 \text{ jumps of } I \text{ in } (0,h))E_\pi[f(x(h),I(h))| \\
&\qquad x(0)=x,I(0)=i,0 \text{ jumps of } I \text{ in } (0,h)]+o(h) \\
&= q_ih\int_0^h f\Bigg(x+\int_0^s \bar\pi(y,i)dy \\
&\quad +\int_s^h \bar\pi(y,1-i)dy)\frac{ds}{h} \\
&\quad +(1-q_ih)f(x+\int_0^h \bar\pi(y,i)dy)+o(h), \\
&\quad \text{where } \bar\pi := \pi - d
\end{aligned}$$

$$
\begin{aligned}
= \quad & q_i f(x, 1-i)h + (1 - q_i h)\left[f(x,i) + \frac{df(\xi,i)}{d\xi}\int_0^h \bar{\pi}(y_i,i)dy\right] \\
& + o(h) \text{ for some} x \leq \xi \leq x + h.
\end{aligned}
$$

Using (29c), we get

$$
\begin{aligned}
& \lim_{h \downarrow 0} \frac{E_\pi[f(x(h), I(h))|x(0) = x, I(0) = i] - f(x,i)}{h} \\
& = \frac{df(x,i)}{dx}[\pi(x,i) - d] + q_i[f(x, 1-i) - f(x,i)].
\end{aligned}
$$

From Dynkin's Formula we have,

$$
\begin{aligned}
& E_\pi\left[\int_0^T \left\{\frac{df(x(t), I(t))}{dx}[\pi(x(t), I(t)) - d]\right.\right. \\
& \left. + q_{I(t)}[f(x(t), 1 - I(t)) - f(x(t), I(t))]\right\} dt \\
& \qquad = E_\pi[f(x(T), I(T)] - f(x(0), I(0)).
\end{aligned}
$$

Now we apply the above formula to W noting that since $|x(t)|$ is bounded by $|x(0)| + (r + d)T$ for $0 \leq t \leq T$ we can modify W outside this range so that it vanishes outside some compact set. So we have

$$
\begin{aligned}
& E_\pi\left[\int_0^T \left\{\frac{dW(x(t), I(t))}{dx}[\pi(x(t), I(t)) - d]\right.\right. \\
& + q_{I(t)}[W(x(t), 1 - I(t)) - W(x(t), I(t))]dt \\
& \qquad = E_\pi[W(x(T), I(T))] - W(x(0), I(0)).
\end{aligned}
$$

From (30a and 30b) it follows that,

$$
\begin{aligned}
E_\pi \int_0^T c(x(t))dt \quad \geq \quad & TJ^* + W(x(0), I(0)) \\
& - E_\pi[W(x(T), I(T))].
\end{aligned}
$$

Dividing by T, taking the limit and using (30c) and (29d) to see that

$$
\frac{E_\pi[W(x(T), I(T))]}{T} \quad \to \quad 0,
$$

yields

$$
\lim \infty_{T \to \infty} \frac{E_\pi \int_0^T c(x(t))dt}{T} \geq J^*.
$$

A similar computation with π^* yields equality.

We need to find $W(\cdot,\cdot)$ and J^* which satisfy the theorem. For this we take advantage of the fact that the average cost can often be obtained as the limit of the discounted cost problem (8), which has been solved in Akella and Kumar. We exclude the trivial case (6) where $(r-d)/q_1 < d/q_0$ and compute the limits $J^* := \lim_{\gamma\downarrow 0} \gamma W_\gamma(x,i)$; $W(x,i) := \lim_{\gamma\downarrow 0}[W_\gamma(x,i) - W_\gamma(z^*,1)]$ and $\pi^*(x,i) := \lim_{\gamma\downarrow 0} \pi_\gamma(x,i)$, where W_γ is the optimal discounted cost obtained in Akella and Kumar and π_γ is the corresponding optimal policy. The results of these computations are summarized in Tables I and II.

Table 1: Computation Summary for Case I:

$$c^+(r-d)(q_0+q_1) - q_1 r(c^+ + c^-) \geq 0$$

$$
\begin{aligned}
J^* &= \frac{c^- q_1 r d}{(q_0+q_1)(q_0 r - q_o d - q_1 d)} \\
W(x) &= \begin{bmatrix} W(x,1) \\ W(x,0) \end{bmatrix} \\
&= \frac{c^-(q_0+q_1)}{2(q_0 r - q_0 d - q_1 d)} x^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{c^- r}{(q_0 r - q_0 d - q_1 d)} x \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&\quad + \frac{c^- r d}{(q_0+q_1)(q_0 r - q_0 d - q_1 d)} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ for } x \leq 0 \\
&= \frac{c^+}{2d} x^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{c^- r d(1 - \frac{e^{-(q_0+q_1)x}}{d})}{(q_0 r - q_0 d - q_1 d)(q_0+q_1)^2} \begin{bmatrix} q_1 \\ -q_0 \end{bmatrix} \\
&\quad - \frac{q_1 r c^-}{(q_0+q_1)(q_0 r - q_0 d - q_1 d)} x \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&\quad + \frac{c^- r d}{(q_0+q_1)(q_0 r - q_0 d - q_1 d)} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ for } x \geq 0 \\
\pi^*(x,i) &= r \text{ if } x 0, i = 1
\end{aligned}
$$

$= d$ if $x = 0, i = 1$

$= 0$ otherwise.

In both Cases I and II it can be verified (the latter being much more tedious) that except possibly for the admissibility and stability of π^*, the given J^*, W, and π^* satisfy (30a, 30b, and 30c). Turning to the verification of admissibility of π^*, the conditions (29a) and (29c) pose no problem, while condition (29b) is proved to hold in Akella and Kumar. Therefore, we need only to prove the stability of π^*, that (29d) holds; then π^* will be proved to be both admissible and stable, and consequently *optimal* through the use of the Verification Theorem.

3 Stability of Optimal Policy

We intend to prove the stability of π^*, i.e., to show that

$$\lim_{T\to\infty} \frac{1}{T} E_{\pi^*}(x^2(T)) \quad = \quad 0. \tag{31}$$

This condition essentially states that the inventory grows more slowly than the square-root of time, and it is clear that any reasonable policy that we wish to implement will have to possess this property. Here, we need to prove this stability property in order to invoke the sufficient conditions for optimality given in the Theorem of the preceding section.

Just for clarity, we focus on the $z^* = 0$ case; the details are similar in the case of $z^* > 0$.

Let $\{\tau_n\}$ be the sequence of stopping times at which the process $(x(t), I(t))$ hits $(0,0)$, that is

$$\begin{aligned} \tau_0 &:= \inf\{t \geq 0 : x(t) = 0, I(t) = 0\}, \\ \tau_{n+1} &:= \inf\{t\tau_n : x(t) = 0, I(t) = 0\}, \end{aligned}$$

(with $\inf \phi := +\infty$). These form *renewal times* for the process (x, I). Let

$$F(s) \quad := \quad Prob(\tau_{n+1} - \tau_n \leq s)$$

be the common probability distribution of the time between successive renewals, and let

$$N(t) \quad := \quad \sup\{n : \tau_n \leq t\}$$

be the number of renewals which have taken place prior to time t. The quantity

$$\delta(t) \quad := \quad t - \tau_{N(t)}$$

is the *current life* which denotes the elapsed time since the last renewal. Since $x(\tau_{N(t)}) = 0$ and the inventory cannot deplete at a rate faster than d, we have

$$0 \geq x(t) \geq -\delta(t)d$$

and so

$$x^2(t) \quad \leq \quad \delta^2(t)d^2.$$

Hence to show (31), it suffices to show that

$$\lim_{T\to\infty} \frac{1}{T} E_{\pi^*}(\delta^2(T)) \quad = \quad 0. \tag{32}$$

We therefore need to examine the second moment of the current life random variable $\delta(T)$.

Fix l and consider a typical renewal period, which is the time interval $[\tau_l, \tau_{l+1}]$. Let

$$\sigma_0 \quad := \quad \tau_l(\text{ note that } I(\sigma_0) = 0) \tag{33}$$
$$\sigma_{m+1} \quad := \quad \inf\{t > \sigma_m : I(t) \neq I(\sigma_m)\} \tag{34}$$

be the successive times at which I switches, and let

$$Y_m := \sigma_{2m-1} - \sigma_{2m-2} \text{ and } Z_m := \sigma_{2m} - \sigma_{2m-1} \tag{35}$$

be, respectively, the lengths of times which I spends in states 0 and 1 before a switch occurs. Note that $\{Y_m\}$ is a sequence of independent and identically exponentially distributed random variables with mean $1/q_0$, while the $\{Z_m\}$ are independent and exponentially distributed with mean $1/q_1$. When $I = 0$ the inventory depletes at rate d; when $I = 1$ the inventory builds up at rate $(r - d)$ (until the inventory returns to 0, whereupon it stays at 0). So

$$x(\sigma_{2m}) \quad = \quad \left(\sum_{i=1}^{m}(r-d)Z_i - dY_i\right) \text{ for } \sigma_{2m} < \tau_l.$$

Hence if

$$\sigma_{2m^*} := \min\left\{\sigma_{2m} : \sum_{i=1}^{m}[(r-d)Z_i - dY_i] \geq 0\right\},$$

then

$$\tau_{l+1} - \tau_l \quad = \quad \min\left\{\sigma_{2m} : \sum_{i=1}^{m}[(r-d)Z_i - dY_i] > 0\right\} + Z_{m^*+1}.$$

Since

$$E[(r-d)Y_i - dZ_i] \quad = \quad \frac{r-d}{q_1} - \frac{d}{q_0} \geq 0,$$

the Law of Large Numbers shows that $\tau_{l+1} < +\infty$ a.s. Hence $F(s)$ is a *proper* distribution.

In what follows, we will assume without loss of generality that $\tau_0 = 0$. Let

$$G_T(s) \quad := \quad Prob(\delta(T) \leq s)$$

be the probability distribution of the current life at time T. From renewal theory (see Karlin and Tyalor, p. 193, 1975) we know that

$$\begin{aligned} G_T(s) \quad &= \quad F(T) - \int_0^{T-s}[1 - F(T-y)]dm(y)sT \\ &= \quad 1 \qquad s \geq T, \end{aligned}$$

where

$$m(t) \quad := \quad E[N(t)]$$

is the mean number of renewals. Hence we may compute the second moment of $\delta(T)$ as follows,

$$\begin{aligned}
& E[\delta^2(T)] \\
&= \int_0^\infty s^2 dG_T(s) \\
&= T^2[1-F(T)] + \int_0^{T-} s^2 dG_T(s) \\
&= T^2[1-F(T)] + \int_0^{T-} \int_0^s 2y dy dG_T(s) \\
&= T^2[1-F(T)] + \int_0^{T-} 2y \int_0^{T-} dG_T(s) dy \\
&= T^2[1-F(T)] + \int_0^T 2y[G_T(T-) - G_T(y)]dy \\
&= T^2[1-F(T)] \\
&\quad + \int_0^T 2y \int_0^{T-y} [1-F(T-z)]dm(z)dy \\
&= T^2[1-F(T)] \\
&\quad + \int_0^T (T-z)^2[1-F(T-z)]dm(z). \qquad (36)
\end{aligned}$$

We now intend to apply the Key Renewal Theorem (see Feller 1968) to obtain the limit of (36) as $T \to \infty$. To do so, we need to show that

$$F \text{ is not lattice.} \qquad (37a)$$

$$\lim_{T\to\infty} T^2[1-F(T)] = 0. \qquad (37b)$$

$$\text{The distribution function } F \text{ has finite mean.} \qquad (37c)$$

$$\text{The function } t^2[1-F(t)] \text{ is directly Riemannintegrable.} \qquad (37d)$$

In the lemma which follows we will prove all four properties. Then, upon applying the Key Renewal Theorem to (36), we will have

$$\lim_{T\to\infty} E_{\pi^*}[\delta^2(T)] \quad = \quad \frac{1}{\mu}\int_0^\infty t^2[1-F(t)]dt < +\infty.$$

So (32), and the stability property (31) will follow readily. To complete this argument, we thus need the following lemma.

Lemma. *The properties (37a-37d) hold.*

Proof: We will prove two facts which are sufficient for the lemma.

$$F(t) \text{ is continuous.} \tag{38a}$$

There exists k and $\delta > 0$ such that

$$1 - F(t) \quad \leq \quad \frac{k}{t^{3+\sigma}} \text{ for all large } t. \tag{38b}$$

Clearly (37a) follows from (38a); (37b) follows from (38b); and (37c) follows from (38b) through the fact that the mean of F can be computed as $\int_0^\infty [1 - F(t)]dt$. Finally, (37a) shows that $t^2[1 - F(t)]$ is Riemann integrable over every finite interval, and (38b) is then a sufficient condition for direct Riemann integrability (see Feller).

To prove both (38a) and (38b), let $\tau_l := 0$ and define $\{\sigma_m, Y_m, Z_m\}$ as in (35). Let $\dot{x}(t) = -d$ for $\sigma_{2m} \leq t\sigma_{2m+1}$ and $\dot{x}(t) = r - d$ for $\sigma_{2m-1} \leq t\sigma_{2m}$, which are, respectively, the intervals in which $I(t) = 0$ and $I(t) = 1$.

For $a \geq 0$, define

$$F(t, a) \quad := \quad Prob(x(s) \geq a \text{ for some } 0 < s \leq t)$$

be the probability distribution of the stopping time when x first rises to a level a. Clearly,

$$F(t, a) \quad = \quad 0 \text{ for } t \leq \frac{a}{r - d},$$

since x cannot rise more rapidly than at rate $(r - d)$. By conditioning on σ_1 and σ_2 we get the renewal equation

$$\begin{aligned} F(t, a) \quad = \quad & \int_0^{rt-dt-a/r} q_0 e^{-q_0\sigma_1} d\sigma_1 \\ & \times \int_{a+\sigma_1 d/r-d}^\infty q_1 e^{-q_1(\sigma_2-\sigma_1)} d(\sigma_2 - \sigma_1) \\ & + \int_0^{rt-dt-a/r} q_0 e^{-q_0\sigma_1} d\sigma_1 \\ & \times \int_0^{(a+\sigma_1 d)/(r-d)} q_1 e^{-q_1(\sigma_2-\sigma_1)} \\ & \times F(t - \sigma_2, a + \sigma_2 d + \sigma_1 r - \sigma_2 r) d(\sigma_2 - \sigma_1). \end{aligned} \tag{39}$$

Let the right hand side above be defined as $(HF)(t, a)$, i.e., the action of an operator H acting on functions $F(x, y)$ with $x \leq t$. Clearly,

$$\sup_{x \leq t_w} |(Hf)(x, w) - (Hg)(x, w)|$$

$$\leq [1 - e^{-q_0 t}] \sup_{\substack{x \leq t \\ w}} |f(x, w) - g(x, w)|,$$

and is therefore a contraction with respect to the sup norm. Moreover, H preserves continuity and so the unique solution F of (39) is continuous. Noting that $F(t)$ is the convolution of $F(t,0)$, with a mean $1/q_1$ exponential distribution, we have proved (38a).

To prove (38b), note that by the Principle of Large Deviations (Varadhan 1982) the quantity

$$1 - P\left(\left|\frac{1}{n}\sum_{i=1}^{n}\left(Y_i - \frac{1}{q_0}\right)\right| \leq \epsilon \right.$$
$$\left. \text{and } \left|\frac{1}{n}\sum_{i=1}^{n}\left(Z_i - \frac{1}{q_1}\right)\right| \leq \delta\right)$$

decreases exponentially in n. This implies that

$$\begin{aligned} 1 - P\Bigg(\sum_{i=1}^{n}(Y_i + Z_i) &\leq n\left(\frac{1}{q_0} + \frac{1}{q_1} + \epsilon + \delta\right) \\ \text{and } \sum_{i=1}^{n}&((r-d)Z_i - dY_i) \\ &\geq n\left[(r-d)\left(\frac{1}{q_1} - \epsilon\right) - d\left(\frac{1}{q_0} + \delta\right)\right]\Bigg) \end{aligned}$$

also decreases exponentially in n. Since

$$\frac{r-d}{q_1} > \frac{d}{q_0},$$

there exist ϵ, $\delta > 0$ so that

$$(r-d)\left(\frac{1}{q_1} - \epsilon\right) - d\left(\frac{1}{q_0} + \delta\right) > 0.$$

With such a choice we see that

$$\begin{aligned} 1 - P\Bigg(\sum_{i=1}^{n}(Y_i + Z_i) &\leq n\left(\frac{1}{q_0} + \frac{1}{q_1} + \epsilon + \delta\right) \\ \text{and } \sum_{i=1}^{n}&((r-d)Z_i - dY_i) > 0\Bigg), \end{aligned}$$

decreases exponentially in n. This implies that so does

$$\left[1 - F\left(n\left(\frac{1}{q_0} + \frac{1}{q_1} + \epsilon + \delta\right)\right)\right],$$

and therefore also $[1 - F(n)]$, proving (38b).

4 CONCLUDING REMARKS

We have shown that zero-inventory policies can be optimal even when there is uncertainty in manufacturing systems. This adds support to the use of such policies, which are currently the focus of much attention due to the greater *discipline* that they enforce on the manufacturing system.

The particular problem for which we have obtained the solution is somewhat restrictive. It would be useful to relax several of the features. For example, the demand rate, which is considered a constant here, may be time-varying and featuring linear growth as well as seasonal and cyclical components. It is also important to take into account the change-over cost when switching production rates. Also, it would be useful to consider more general batch manufacturing problems.[1]

In a different direction, it is of much interest to examine the complex model of a multi-machine, multi-product flexible manufacturing system featured in Kimemia and Gershwin (1983) to show that there are ranges of mean times between failures and mean repair times under which zero-inventory policies remain optimal.

ACKNOWLEDGEMENTS

The authors are grateful to the editor, associate editor and the two referees for a number of constructive suggestions including those in Section **??**. The research of the second author has been supported by the U.S. Army Research Office under Contract No. DAAG-29-85-K0094, and by the Joint Services Electronics Program under Contract No. NOOO14-85-C0149. The work of the first author was supported by a Fulbright Scholarship while he was visiting the University of Illinois.

REFERENCES

Akella, R., and P. R. Kumar. 1986. Optimal Control of Production Rate in a Failure Prone Manufacturing System. *IEEE Trans. on Automatic Control*, AC-31, **2**, 116-126.

Feller, W. 1968. *An Introduction to Probability Theory and Its Applications*, Vol. II, John Wiley & Sons, New York.

Hall, R. W. 1983. *Zero Inventories*, Dow Jones-Irwin Press. Homewood, Ill.

Karlin, S., and H. M. Taylor. 1975. *A First Course in Stochastic Processes*, Academic Press, New York.

[1] The authors are grateful to a reviewer for these suggestions.

Karlin, S., and H. M. Taylor. 1981. *A Second Course in Stochastic Processes*, Academic Press, New York.

Kimemia, J. G., and S. B. Gershwin. 1983. An Algorithm for the Computer Control of Production in Flexible Manufacturing Systems. *IEE Transactions*, **15**, 353–362.

Varadhan, S. R. S. 1982. *Large Deviations and Applications*, SIAM-CBMS Press, Philadelphia.

Table 2: Computation Summary for Case II: $c^+(r-d)(q_0+q_1)-q_1 r(c^+ + c^-) \leq 0$

$$
\begin{aligned}
J^* &= \frac{c^+ d}{q_0+q_1} + \frac{c^+}{(\frac{q_0}{d} - \frac{q_1}{r-d})} \log[\frac{r q_1 (c^+ + c^-)}{c^+(r-d)(q_0+q_1)}] \\
W(x) &= \begin{bmatrix} W(x,1) \\ W(x,0) \end{bmatrix} \\
&= c^+ \left(\sum_{n=1}^{\infty} \frac{A_1^{n-1}(-z^*)^n}{n!} \right) \left(\sum_{m=1}^{\infty} \frac{A_1^{m-1} x^m}{m!} \right) b_1 - \frac{c^+}{q_1} e^{(-A_1 z)} \left(\sum_{n=1}^{\infty} \frac{A_1^{n-1} x^n}{n!} \right) [\\
&\quad -c^- \left(\sum_{n=2}^{\infty} \frac{A_1^{n-2} x^n}{n!} \right) b_1 + \left(\sum_{n=2}^{\infty} \frac{A_1^{n-2}(-z^*)^n}{n!} \right) \left(c^+ b_1 - \frac{c^+}{q_1} A_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) + \left(c^+ \right. \\
&= \left(\sum_{n=2}^{\infty} \frac{A_1^{n-2}(x-z^*)^n}{n!} \right) \left(c^+ b_1 - \frac{c^+}{q_1} A_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \\
&\quad - \left(\frac{c^+(x-z^*)}{q_1} - \frac{c^+ d}{q_1(q_0+q_1)} \right) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ for } 0 \leq\leq z^* \\
&= c^+ \left(\sum_{n=2}^{\infty} \frac{A_2^{n-2}(x-z^*)^n}{n!} \right) b_2 - \frac{c^+}{q_1} \left(\sum_{n=1}^{\infty} \frac{A_2^{n-1}(x-z^*)^n}{n!} \right) \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\end{aligned}
$$

$$
-\frac{c^+ d}{q_1(q_0+q_1)} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ for } x \geq z^*
$$

where

$$
A_1 = \begin{bmatrix} \frac{q_1}{r-d} & \frac{-q_1}{r-d} \\ \frac{-q_0}{d} & \frac{q_0}{d} \end{bmatrix}; A_2 = \begin{bmatrix} \frac{-q_1}{d} & \frac{q_1}{d} \\ \frac{q_0}{d} & \frac{-q_0}{d} \end{bmatrix}; b_1 = \begin{bmatrix} \frac{-1}{r-d} \\ \frac{1}{d} \end{bmatrix}; b_2 = \begin{bmatrix} \frac{1}{d} \\ \frac{1}{d} \end{bmatrix}
$$

and

$$
z^* = \frac{1}{(\frac{q_0}{d} - \frac{q_1}{r-d})} \log[\frac{r q_1 (c^+ + c^-)}{c^+(r-d)(q_0+q_1)}].
$$

$$
\begin{aligned}
\pi^*(x,i) &= r if x z^*, i = 1 \\
&= d if x = z^*, i = 1 \\
&= 0 \text{ otherwise.}
\end{aligned}
$$