On the Capacity Requirement of Largest-Deficit-First for Scheduling Real-Time Traffic in Wireless Networks

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ABSTRACT

We consider ad hoc wireless networks with real-time traffic, and study the capacity requirement of a low-complexity scheduling policy, called largest-deficit-first (LDF), for achieving the same quality of service (QoS) as the optimal policy in networks with unit capacity. We derive theoretical upper and lower bounds for general traffic. The bounds depend on the interference degree of the network and the max/min delay bound ratio. The performance of LDF is further evaluated using simulations and compared with two other algorithms. The simulation results show that LDF significantly outperforms the other two algorithms.

Categories and Subject Descriptors

C.2.1 [**Computer-Communication Networks**]: Network Architecture and Design—*wireless communication*

General Terms

Theory

Keywords

Capacity requirement; largest-deficit-first; real-time scheduling; quality of service; ad hoc networks

1. INTRODUCTION

With the rapid rise of various real-time applications such as video streaming, a large portion of the traffic in wireless networks is delay sensitive. For this kind of real-time traffic, there has been great interest in providing both high network throughput and minimum quality of service (QoS) guarantees for individual links. Following the analytical framework for scheduling traffic with hard deadlines proposed by Hou et al. [3], a lot of effort has been put in understanding the QoS performance of different scheduling policies with general network topology, heterogeneous traffic, channel fading etc. [4, 5, 6, 8] Other approaches have also been adopted to study QoS in multihop real-time traffic scenario. [13, 14] However, the problem of scheduling packets with hard deadlines in ad hoc wireless networks is largely unsolved except for some special cases as in Jaramillo et al. [9], where packet arrivals and deadline information are available before the packets arrive. Even with the required future information, the complexity of the algorithm in Jaramillo et al. [9] is prohibitively high. Because of that, Kang et al. [11] have studied the performance of a low-complexity algorithm, called the largest-deficit-first (LDF) scheduling, which maintains a deficit for each link and greedily selects links with the largest deficits,[1] and quantified the achievable delivery ratio of LDF compared to the optimal policy in the same network, which is called the *QoS efficiency ratio* in this paper. The QoS efficiency ratio quantifies the difference in delivery ratio between a greedy algorithm and the optimal algorithm. In particular, the QoS efficiency ratio is said to be γ if the number of successfully delivered packets under the greedy algorithm is at least γ faction of that under the optimal algorithm for any flow. This paper studies the performance of LDF from a different perspective and focuses on the amount of additional capacity required to *achieve the same delivery ratio as the optimal policy in networks with unit capacity*, which is named the *capacity efficiency ratio*.

We remark that these two concepts of efficiency ratios are very different because an increase in channel capacity does not necessarily result in the same amount of increase in delivery ratio. This can be easily seen from the following simple example. Consider a wireless channel with unit capacity and Poisson-distributed arrivals with rate λ. Assume the system is time-slotted; every packet is associated with a hard deadline of one time slot so a packet is either transmitted within the same time at which it arrives or discarded. In this simple example, the effective throughput of the system is

$$\mathbb{P}\left(A(t) \geq 1\right) = 1 - \frac{\lambda^0 e^{-\lambda}}{0!} = 1 - e^{-\lambda},$$

where $A(t)$ is the number of packet arrivals at the beginning of time slot t, and the delivery ratio is

$$(1 - e^{-\lambda})/\lambda.$$

Now if the channel capacity increases to two packets per time slot, then the effective throughput is

$$2\,\mathbb{P}\left(A(t) \geq 2\right) + \mathbb{P}\left(A(t) = 1\right) = 2 - (2 + \lambda)e^{-\lambda},$$

[1] A formal definition of LDF will be given in Section 2.3.

MobiHoc'15, June 22–25, 2015, Hangzhou, China.

ACM 978-1-4503-3489-1/15/06 ...$15.00.
http://dx.doi.org/10.1145/2746285.2746302.

and the delivery ratio becomes

$$(2-(1+2\lambda)e^{-\lambda})/\lambda.$$

So when the bandwidth doubles, the increase in effective throughput or delivery ratio is less than 2-fold if $\lambda > 0.5$ (e.g., the increase is about $1.53 < 2$ when $\lambda = 2$). This is mainly because of the hard deadline constraint associated with each packet. If no deadline is imposed and $\lambda > 2$, then the effective throughput doubles when the channel capacity increases from one packet per time slot to two packets per time slot. We will later see in this paper that the bounds on the capacity efficiency ratio are completely different from the bounds on the QoS efficiency ratio.

We further comment that in Kang et al. [11], the authors proved that the QoS efficiency ratio of LDF is lower-bounded by $1/(1+\beta)$, where β is the interference degree (see Chaporkar et al. [1]) of the network, i.e., the maximum number of links interfering with a single link that can be simultaneously activated in the network. In other words, the delivery ratio guaranteed by LDF is at least $1/(1+\beta)$ of the delivery ratio of the optimal policy. Since $\beta \geq 1$, the results in Kang et al. [11] can only be used to study the performance of LDF for those scenarios where the required delivery ratios are less than 0.5, and do not address the efficiency of LDF for applications which require higher delivery ratios. Therefore, to understand the performance of LDF for supporting high delivery ratios, we study the capacity efficiency ratio in this paper. In other words, the goal is to quantify the amount of additional capacity required for a greedy algorithm to achieve the same delivery ratios as the optimal policy for given traffic, where the delivery ratios can be close to one. The results in this paper are fundamentally different from the results in [11] as the results in [11] can only be applied to scenarios and applications where low delivery ratios are sufficient.

In this paper, we consider general network topology and traffic process, and derive theoretical bounds on the capacity efficiency ratio for LDF. Recall that the capacity efficiency ratio, denoted by K_{LDF}, is the minimum capacity required by LDF to guarantee at least the same QoS region achieved by the optimal policy with unit capacity.[2] We establish the following upper and lower bounds on K_{LDF}, where $\frac{\tau_{\max}}{\tau_{\min}}$ is the ratio of maximum packet delay bound to minimum packet delay bound, and β is the aforementioned interference degree of the network.

1. For i.i.d. traffic,

$$K_{\text{LDF}} \leq \left(2\frac{\tau_{\max}}{\tau_{\min}} + 2\log\frac{\tau_{\max}}{\tau_{\min}} + 3\right)\beta.$$

2. For i.i.d. traffic with unit delay bound ($\tau_{\min} = \tau_{\max} = 1$),

$$K_{\text{LDF}} \leq 2\beta.$$

3. There is non-i.i.d. probabilistic adversary such that

$$K_{\text{LDF}} \geq \frac{\tau_{\max}}{\tau_{\min}}.$$

We see from these bounds that to match the optimal but unknown policy with unit capacity, the required capacity for LDF depends on not only the topology characteristic but also the packet delay bound variation. While the upper and lower bounds are obtained under different assumptions, these bounds, along with the simulation results to be provided between LDF and other scheduling policies, provide basic understanding of the capacity requirements of LDF for supporting deadline-constrained traffic flows from various aspects of this problem.

The remaining of this paper is organized as follows. Section 2 introduces the model of the problem and defines the capacity efficiency ratio for capacity requirement characterization. We then present our main results regarding upper and lower bounds on the capacity efficiency ratio of LDF in Section 3. Simulation results are provided in Section 4. Section 5 concludes the paper.

[2]The rigorous definition of the capacity efficiency ratio will be given in Section 2.4.

2. MODEL

In this section we introduce the system model for the real-time scheduling problem. The wireless ad hoc network is modeled by a graph with potentially complex topology. The real-time data traffic is modeled to be (possibly infinitely many) packets with stochastic arrival times and hard deadlines. Then we describe the possible scheduling policies in a system with a certain capacity, and formally present the capacity efficiency ratio, which is the metric we use to characterize the performance of a scheduling policy.

We use $(x_i : i \in I)$ to denote the family (vector) of elements indexed by I. Curly braces with colons are reserved for set-builders. The size (cardinality) of a set A is denoted by $|A|$.

2.1 Network Model

We consider a time-slotted system with the set of links $\mathcal{N} = \{1, 2, \ldots, N\}$. The interference among the links is characterized by a conflict graph $G = (\mathcal{N}, \mathcal{E})$, where $\mathcal{N}$ is the set of vertices, each vertex corresponding to a link in the original network, and there is an edge (i_1, i_2) in $\mathcal{E}$ if link i_1 and link i_2 interfere with each other and thus cannot be scheduled simultaneously. Then $\mathcal{L} \subseteq \mathcal{N}$ is an independent set of the conflict graph if for any i_1, $i_2 \in \mathcal{L}$, we have $(i_1, i_2) \notin \mathcal{E}$. Let $\mathcal{I}$ be the set of all the independent sets of G. We then define the set of neighbors of link i to be those links that interfere with i, denoted by $\text{NB}(i) = \{i' \in \mathcal{N} : (i, i') \in \mathcal{E}\}$. Then the interference degree of G is given by

$$\beta = \max\{|\mathcal{L}| : i \in \mathcal{N}, \mathcal{L} \in \mathcal{I}, \mathcal{L} \subseteq \text{NB}(i)\}.$$

2.2 Traffic Model

At each time slot a number of packets with hard deadlines arrive on the links. The traffic of packets arriving in the system can be specified by a traffic sample path $J = ((n_i, b_i, e_i) : i \in \mathcal{N})$ with $b_i = (b_i(j) : 1 \leq j \leq n_i)$ and $e_i = (e_i(j) : 1 \leq j \leq n_i)$, where $n_i \leq \infty$ is the total number of packets arriving on link i in this traffic sample path, and $b_i(j)$ and $e_i(j)$ are the arriving time and the deadline of the jth packet on link i, respectively. Note that $n_i < \infty$ indicates that there are finitely many packets arriving on link i in the traffic sample path J, while $n_i = \infty$ indicates that there are infinitely many such packets in the traffic sample path J. Since the deadline is never before the arriving time, we have $b_i(j) \leq e_i(j)$ for all i and all j. The delay bound of the jth packet on link i, which is the maximum delay from the arriving time to the time it is scheduled, is

$$\tau_i(j) = e_i(j) - b_i(j) + 1.$$

If the packet j on link i has not been scheduled by the end of time slot $e_i(j)$, it is considered useless and is discarded from the system. Let the number of arriving packets in J on link i at time slot t be $a_i(t) = |\{j: 1 \le j \le n_i, b_i(j) = t\}|$. By assuming the number of arriving packets on each link per time slot is at most $a_{\max}$ we have $a_i(t) \le a_{\max}$ for any link i and any time t. Let $\mathcal{J}$ be the set of all traffic sample paths with delay bounds between $\tau_{\min}$ and $\tau_{\max}$ and maximum number of arriving packets $a_{\max}$.

2.3 Scheduling Policy

A scheduling policy determines a set of packets to be scheduled at each time slot. We assume reliable channels, so once scheduled a packet will be successfully delivered and leave the system. Let the set of available packets on link i at the beginning of time slot t be $\mathcal{A}_i(t) \subseteq \{1, 2, \ldots, n_i\}$. Then for any $j \in \mathcal{A}_i(t)$ we have $b_i(j) \le t \le e_i(j)$. Let the set of packets scheduled by a scheduling policy on link i at time slot t be $\mathcal{S}_i(t) \subseteq \mathcal{A}_i(t)$. For a system with unit capacity, at most one packet can be scheduled on a link each time slot (i.e., $|\mathcal{S}_i(t)| \le 1$), and the set of links on which packets are scheduled must be an independent set of the conflict graph G due to the interference constraint (i.e., $\{i \in \mathcal{N}: |\mathcal{S}_i(t)| > 0\} \in \mathcal{I}$).

We now define the capacity of a system. We say a system has capacity k if each time slot is divided into k sub-slots, and the scheduling policy can choose a link activation set in each of the k sub-slots sequentially in a single time slot (this is the case for wireless networks where multiple orthogonal channels can be used for transmission).[3] Formally, given the packet$(\mathcal{A}_i(t): i \in \mathcal{N})$ at time t, the set of scheduled packets on link i is

$$\mathcal{S}_i(t) = \bigcup_{k'=1}^{k} \mathcal{S}_i^{k'}(t),$$

where the sub-slot schedules $\left(\mathcal{S}_i^{k'}(t): i \in \mathcal{N}, 1 \le k' \le k\right)$ satisfy the following constraints:

1. Only available packets can be scheduled; i.e.,

$$\mathcal{S}_i^{k'}(t) \subseteq \mathcal{A}_i(t) \quad \forall k' \le k.$$

2. Each sub-slot of schedule has unit capacity; i.e.,

$$\left|\mathcal{S}_i^{k'}(t)\right| \le 1 \quad \forall k' \le k.$$

3. No packets can be (or need to be) scheduled twice since schedules in different sub-slots of a time slot are determined sequentially; i.e.,

$$\mathcal{S}_i^{k_1}(t) \cap \mathcal{S}_i^{k_2}(t) = \emptyset \quad \forall k_1, k_2 \le k \text{ with } k_1 \ne k_2.$$

4. In each sub-slot the schedule satisfies the interference constraint; i.e.,

$$\left\{i \in \mathcal{N}: \left|\mathcal{S}_i^{k'}(t)\right| > 0\right\} \in \mathcal{I} \quad \forall k' \le k.$$

[3]In this paper we consider uniform capacity over all the links in the network, which is the case when a particular wireless communication protocol is adopted in the ad hod network. E.g., IEEE 802.11a uses 52 subcarriers, which can be interpreted as a uniform capacity of 52 throughout the whole network.

The system, or the scheduling policy, is said to have unit capacity if it has capacity 1.

Naturally, we study maximal scheduling policies which do not idle when there is an available packet on a link that does not interfere with any other scheduled link. Note that some maximal policies are not feasible in practice since their decisions may depend on future traffic information.

We focus on a low-complexity greedy scheduling policy called the largest-deficit-first policy (LDF), which is the real-time variation of the extensively-studied longest-queue-first scheduling policy (see, e.g., Dimakis and Walrand [2] and Le et al. [12]). LDF maintains a nonnegative virtual queue $d_i(t)$, called the *deficit*, for each link i. The deficit evolves as follows.

1. The initial deficit is $d_i(0) = 0$ for all $i \in \mathcal{N}$.

2. Each time a packet arrives on link i, the deficit $d_i(t)$ increases by one with probability q_i, where q_i is the required delivery ratio for link i.[4]

3. Each time a packet on link i is scheduled, the deficit $d_i(t)$ decreases by one deterministically if $d_i(t) > 0$, and remain unchanged if $d_i(t) = 0$.

We can see that the deficit on a link is a nonnegative integer keeping track of the number of packets the scheduler owes to the link in order to meet the required delivery ratio, and when the deficit remains small, the required delivery ratio is achieved. In each sub-slot of the time slot t, LDF first sorts the links in $\mathcal{N}$ according to the current deficits $d_i(t)$, and obtains the index vector I such that $d_{I_1}(t) \ge d_{I_2}(t) \ge \cdots \ge d_{I_N}(t)$. LDF starts with the set of a single link that has the largest deficit, namely $\mathcal{B} = \{I_1\}$, and repeatedly considers link $i = 2, 3, \ldots, N$. The link I_i is included in the selection $\mathcal{B}$ if the following hold.

1. Link I_i does not interfere with any link in $\mathcal{B}$.

2. Link I_i has available packets on it for transmission.

After all links have been considered, the selection $\mathcal{B}$ is the scheduling decision given by LDF at this sub-slot of the time slot t.

2.4 Delivery Ratio Characterization

The goal of this paper is to provide individual quality-of-service (QoS) guarantee, which is usually more demanding than total throughput guarantee since some links may be over-served (getting more service than necessary) while others starved (getting insufficient service). In other words, the required delivery ratio for each link may not be achieved. That motivates us to study the relation between delivery ratio and capacity.

We model the stochastic traffic by the traffic process X, which is a $\mathcal{J}$-valued random element. We assume that LDF is associated with an arbitrary random tie-breaking process. We say the delivery ratio vector $q \in [0,1]^N$ is *achieved* by $\text{LDF}^{[k]}$ (LDF with capacity k) if for any $i \in \mathcal{N}$,

$$\mathbb{P}\left(\liminf_{T\to\infty} \frac{\text{LDF}_i^{[k]}(X,T)}{\text{CA}_i(X,T)} \ge q_i\right) = 1,$$

[4]We will elaborate on the meaning of the required delivery ratio q_i in Section 2.4.

where $\mathrm{LDF}_i^{[k]}(X,T)$ is the number of packets in X on link i scheduled by $\mathrm{LDF}^{[k]}$ in the first T time slots, and $\mathrm{CA}_i(X,T)$ is the cumulative number of arriving packets in traffic process X on link i before time T. Formally, for $X=((n_i,b_i,e_i)\colon i\in\mathcal{N})$,

$$\mathrm{CA}_i(X,T)=\left|\{j\colon 1\le j\le n_i, b_i(j)\le T\}\right|.$$

Let the delivery ratio stability region of $\mathrm{LDF}^{[k]}$ for X be

$$\Lambda_{\mathrm{LDF}^{[k]}}=\left\{q\in[0,1]^N\colon q \text{ is achieved by } \mathrm{LDF}^{[k]}\right\}.$$

Define the delivery ratio stability region for traffic process X, denoted by Λ, to be the set of delivery ratios that can be achieved by some maximal policy with unit capacity. We then define the *capacity efficiency ratio* of LDF for traffic process X to be

$$K_{\mathrm{LDF}}=\min\left\{k\colon\ \mathrm{int}(\Lambda)\subseteq\Lambda_{\mathrm{LDF}^{[k]}}\right\},$$

where $\mathrm{int}(\cdot)$ is the interior. This definition is well-defined since sufficiently large capacity can guarantee that all packets will be delivered by any maximal scheduling policy when the number of arriving packets in each time slot is bounded.

3. MAIN RESULTS

We present upper and lower bounds on the capacity efficiency ratio of LDF in this section.

THEOREM 1. *For traffic process X that has i.i.d. arrival across time slots and i.i.d. delay bound across different packets on a link,*

$$K_{\mathrm{LDF}}\le\left(2\frac{\tau_{\max}}{\tau_{\min}}+2\log\frac{\tau_{\max}}{\tau_{\min}}+3\right)\beta.$$

We first state a lemma that gives an upper bound on the capacity efficiency ratio regardless of the stochastic nature of the traffic.

LEMMA 1. *For i.i.d. traffic process, the capacity efficiency ratio of LDF is upper-bounded by the minimum capacity that guarantees a maximal scheduling policy outperforms any unit-capacity policy in throughput over all traffic sample paths. Formally,*

$$\begin{aligned}K_{\mathrm{LDF}}\le\min\{k\colon &\mu^{[k]}(J,T)\ge\nu^{[1]}(J,T)\\ &\forall J\in\mathcal{J},\forall T,\forall\mu^{[k]}\in\mathcal{P}(k),\forall\nu^{[1]}\in\mathcal{P}(1)\},\end{aligned}$$

where $\mathcal{P}(k)$ and $\mathcal{P}(1)$ are the sets of all maximal scheduling policies with capacities k and 1, respectively, and $\mu^{[k]}(J,T)$ and $\nu^{[1]}(J,T)$ are the numbers of packets in traffic sample path J scheduled by $\mu^{[k]}$ and $\nu^{[1]}$ in the first T time slots.

Note that, by the definition of $\mathcal{P}(k)$ and $\mathcal{P}(1)$, $\mu^{[k]}$ and $\nu^{[1]}$ are two arbitrary maximal scheduling policies with capacities k and 1. So the right-hand-side of Lemma 1 is for the worst-case traffic sample paths and thus does not depend on the (probability) law of the traffic process. The proof of Lemma 1 can be found in Appendix A.

PROOF OF THEOREM 1. For any finite traffic sample path $J\in\mathcal{J}$ with $n_i<\infty$ for all $i\in\mathcal{N}$, let $\mathcal{L}^{\mathrm{P}}$ be the set of all packets on the links. For any $j\in\mathcal{L}^{\mathrm{P}}$, let $l(j)$ be the link of packet j, $b(j)$ the arriving time, $e(j)$ the deadline, and $\tau(j)=: e(j)-b(j)+1$ the delay bound. We can then construct a conflict graph of the packets $G^{\mathrm{P}}=(\mathcal{L}^{\mathrm{P}},\mathcal{E}^{\mathrm{P}})$, where there is an edge between $j\in\mathcal{L}^{\mathrm{P}}$ and $j'\in\mathcal{L}^{\mathrm{P}}$ if either $(l(j),l(j'))\in\mathcal{E}$ or $l(j)=l(j')$.

Let $\mathcal{O}\subseteq\mathcal{L}^{\mathrm{P}}$ be the set of packets scheduled by some policy ν with unit capacity, and let $\mathcal{M}\subseteq\mathcal{L}^{\mathrm{P}}$ be the set of packets scheduled by $\mathrm{MAX}^{[k]}$ (an arbitrary maximal scheduling policy with capacity k). By Lemma 1, to show the upper bound we only need to show that if $k=\left(2\frac{\tau_{\max}}{\tau_{\min}}+2\log\frac{\tau_{\max}}{\tau_{\min}}+3\right)\beta$ then $|\mathcal{M}|\ge|\mathcal{O}|$. Let t_j^{M} and t_j^{O} be the time slot that packet j is scheduled by $\mathrm{MAX}^{[k]}$ and ν respectively, and $t_j^{\mathrm{M}}=-1$ or $t_j^{\mathrm{O}}=-1$ if j is not scheduled by $\mathrm{MAX}^{[k]}$ or ν. For any $j\notin\mathcal{M}$, let

$$B_j:=\left\{j'\in\mathcal{M}\colon b(j)\le t_{j'}^{\mathrm{M}}\le e(j),(j,j')\in\mathcal{E}^{\mathrm{P}}\right\}.$$

Then B_j can be understood as the potential set of blockers of j in $\mathcal{M}$, which are the packets that interfere with packet j and are are scheduled by $\mathrm{MAX}^{[k]}$ at a feasible scheduling time of j. Define the weights

$$w_j:=\begin{cases}1 & \text{if } j\in\mathcal{O}\cap\mathcal{M}\\ \frac{|B_j\backslash\mathcal{O}|}{(2+2\log(\tau_{\max}/\tau_{\min}))\beta\tau(j)} & \text{if } j\in\mathcal{O}\backslash\mathcal{M}\\ 0 & \text{if } j\notin\mathcal{O}\end{cases}.$$

To show $|\mathcal{M}|\ge|\mathcal{O}|$ we only need to show the following two inequalities:

$$\sum_{j\in\mathcal{L}^{\mathrm{P}}}w_j\le|\mathcal{M}|,\tag{1}$$

and

$$|\mathcal{O}|\le\sum_{j\in\mathcal{L}^{\mathrm{P}}}w_j.\tag{2}$$

To see (1) holds, we notice

$$\begin{aligned}&\sum_{j\in\mathcal{L}^{\mathrm{P}}}w_j-|\mathcal{M}|\\ &=\sum_{j\in\mathcal{O}}w_j-\sum_{j'\in\mathcal{M}}1\\ &=\sum_{j\in\mathcal{O}\backslash\mathcal{M}}\frac{|B_j\backslash\mathcal{O}|}{(2+2\log(\tau_{\max}/\tau_{\min}))\beta\tau(j)}-\sum_{j'\in\mathcal{M}\backslash\mathcal{O}}1\\ &=\sum_{j'\in\mathcal{M}\backslash\mathcal{O}}\left(\sum_{j\in\mathcal{O}\backslash\mathcal{M}}\frac{I_{B_j\backslash\mathcal{O}}(j')}{(2+2\log(\tau_{\max}/\tau_{\min}))\beta\tau(j)}-1\right),\end{aligned}$$

where

$$I_{B_j\backslash\mathcal{O}}(j')=\begin{cases}1 & \text{if } j'\in B_j\backslash\mathcal{O}\\ 0 & \text{otherwise}\end{cases}.$$

Note in the above we rearranged the weights on $\mathcal{O}\backslash\mathcal{M}$ by the blockers in $\mathcal{M}\backslash\mathcal{O}$. By the definition of interference degree β, in each time slot there are at most β "neighbors" of $j'\in\mathcal{M}$ that are scheduled by ν. So for any τ with $\tau_{\min}\le\tau\le\tau_{\max}$,

$$\left|\left\{j\in\mathcal{O}\colon j'\in B_j,\tau(j)\le\tau\right\}\right|\le(2\tau-1)\beta.\tag{3}$$

We can see this in Figure 1, where packet j' can block at most $(2\tau-1)\beta$ packets in $\mathcal{O}$ with delay bounds less than or equal to τ. Hence,

$$\sum_{j\in\mathcal{O}\backslash\mathcal{M}}\frac{I_{B_j\backslash\mathcal{O}}(j')}{\tau(j)}$$

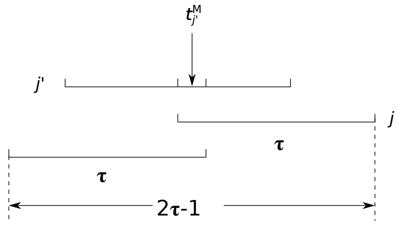

Figure 1: Demonstration of inequality (3).

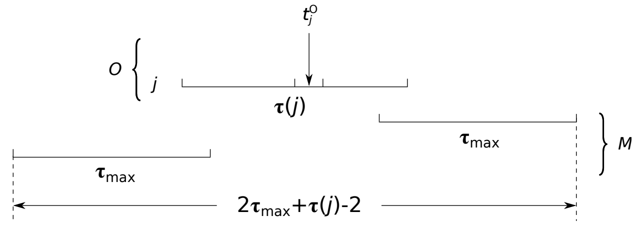

Figure 2: Demonstration of inequality (4).

$$
\begin{aligned}
&= \sum_{\tau=\tau_{\min}}^{\tau_{\max}} \sum_{\substack{j\in\mathcal{O}\backslash\mathcal{M}\\ \tau(j)=\tau}} \frac{I_{B_j\backslash\mathcal{O}}(j')}{\tau(j)} \\
&= \sum_{\tau=\tau_{\min}}^{\tau_{\max}} \frac{1}{\tau} \sum_{\substack{j\in\mathcal{O}\backslash\mathcal{M}\\ \tau(j)=\tau}} I_{B_j\backslash\mathcal{O}}(j') \\
&\le \sum_{\tau=\tau_{\min}}^{\tau_{\max}} \frac{1}{\tau} \left|\{j\in\mathcal{O}\colon j'\in B_j, \tau(j)=\tau\}\right| \\
&= \sum_{\tau=\tau_{\min}}^{\tau_{\max}} \frac{1}{\tau}(\left|\{j\in\mathcal{O}\colon j'\in B_j, \tau(j)\le\tau\}\right| \\
&\qquad - \left|\{j\in\mathcal{O}\colon j'\in B_j, \tau(j)\le\tau-1\}\right|) \\
&= \sum_{\tau=\tau_{\min}}^{\tau_{\max}-1} \left|\{j\in\mathcal{O}\colon j'\in B_j, \tau(j)\le\tau\}\right| \left(\frac{1}{\tau}-\frac{1}{\tau+1}\right) \\
&\quad + \left|\{j\in\mathcal{O}\colon j'\in B_j, \tau(j)\le\tau_{\max}\}\right| \frac{1}{\tau_{\max}} \\
&\le \sum_{\tau=\tau_{\min}}^{\tau_{\max}-1} (2\tau-1)\beta\left(\frac{1}{\tau}-\frac{1}{\tau+1}\right) + (2\tau_{\max}-1)\beta\frac{1}{\tau_{\max}} \\
&\le \left(\frac{2\tau_{\min}-1}{\tau_{\min}} + \frac{2}{\tau_{\min}+1} + \cdots + \frac{2}{\tau_{\max}}\right)\beta \\
&\le \left(2+2\log\frac{\tau_{\max}}{\tau_{\min}}\right)\beta.
\end{aligned}
$$

We then get

$$\sum_{j\in\mathcal{L}^{\mathrm{P}}} w_j - |\mathcal{M}| \le \sum_{j'\in\mathcal{M}\backslash\mathcal{O}} (1-1) = 0.$$

This completes the proof of (1).

Since $\mathrm{MAX}^{[k]}$ has capacity k, any $j\in\mathcal{O}\backslash\mathcal{M}$ must have been blocked by at least $k\tau(j)$ packets in $\mathcal{M}$. I.e.,

$$|B_j| \ge k\tau(j).$$

We call $B_j\cap\mathcal{O}$ the set of type I blockers of j, and $B_j\backslash\mathcal{O}$ the set of type II blockers of j. From Figure 2 we see that the type I blockers of j can be scheduled by ν in the $2\tau_{\max}+\tau(j)-2$ time slots except time slot t_j^{O}. By the interference degree, we must have

$$|B_j\cap\mathcal{O}| \le (2\tau_{\max}+\tau(j)-3)\beta. \qquad (4)$$

Since j is blocked by some packet $j'\in\mathcal{M}\backslash\mathcal{O}$ every sub-slot within the delay bound of j, we have that the number of total blockers of packet j is at least the capacity k. So we can get a lower bound on the number of type II blockers of packet j as follow.

$$
\begin{aligned}
|B_j\backslash\mathcal{O}| &= |B_j| - |B_j\cap\mathcal{O}| \\
&\ge k\beta\tau(j) - (2\tau_{\max}+\tau(j)-3)\beta \\
&= \left(2\frac{\tau_{\max}}{\tau_{\min}} + 2\log\frac{\tau_{\max}}{\tau_{\min}} + 3\right)\beta\tau(j) \\
&\quad - (2\tau_{\max}+\tau(j)-3)\beta \\
&\ge \left(2+2\log\frac{\tau_{\max}}{\tau_{\min}}\right)\beta\tau(j).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\sum_{j\in\mathcal{L}^{\mathrm{P}}} w_j - |\mathcal{O}| &= \sum_{j\in\mathcal{O}} (w_j - 1) \\
&= \sum_{j\in\mathcal{O}\backslash\mathcal{M}} \left(\frac{|B_j\backslash\mathcal{O}|}{\left(2+2\log\frac{\tau_{\max}}{\tau_{\min}}\right)\beta\tau(j)} - 1\right) \\
&\ge \sum_{j\in\mathcal{O}\backslash\mathcal{M}} (1-1) \\
&\ge 0.
\end{aligned}
$$

Thus (2) is proved. Combining (1), (2) and Lemma 1 we get Theorem 1. □

THEOREM 2. *For i.i.d. traffic process X with unit delay bound,*

$$K_{\mathrm{LDF}} \le 2\beta.$$

REMARK 1. *We note that Theorem 2 is stronger than Theorem 1 with i.i.d. traffic and $\tau_{\max}=\tau_{\min}=1$, which only gives an upper bound of 5β.*

PROOF. As in the proof of Theorem 1, we consider the packet conflict graph $G^{\mathrm{P}} = (\mathcal{L}^{\mathrm{P}}, \mathcal{E}^{\mathrm{P}})$, where $\mathcal{L}^{\mathrm{P}}$ is the set of packets. Since the traffic has unit delay bound, we only need to consider one set of arriving packets $\mathcal{L}^{\mathrm{P}}$ with $b(j) = e(j) = 1$ for all $j\in\mathcal{L}^{\mathrm{P}}$. Let

$$B_j := \left\{j'\in\mathcal{M}\colon (j,j')\in\mathcal{E}^{\mathrm{P}}\right\}.$$

We use the following weights:

$$w_j := \begin{cases} 1 & \text{if } j\in\mathcal{O}\cap\mathcal{M} \\ \frac{|B_j\backslash\mathcal{O}|}{\beta} & \text{if } j\in\mathcal{O}\backslash\mathcal{M} \\ 0 & \text{if } j\notin\mathcal{O} \end{cases}.$$

Similarly we can see that since the interference degree is β, we have

$$\sum_{j\in\mathcal{O}\backslash\mathcal{M}} I_{B_j\backslash\mathcal{O}}(j') \le \beta,$$

and then

$$\sum_{j\in\mathcal{L}^{\mathrm{P}}} w_j - |\mathcal{M}| = \sum_{j'\in\mathcal{M}\backslash\mathcal{O}} \left(\sum_{j\in\mathcal{O}\backslash\mathcal{M}} \frac{I_{B_j\backslash\mathcal{O}}(j')}{\beta} - 1\right) \le 0.$$

So (1) still holds. By choosing $k = 2\beta$ we have

$$|B_j \backslash \mathcal{O}| = |B_j| - |B_j \cap \mathcal{O}| \geq k - \beta = \beta.$$

So $\sum_{j \in \mathcal{L}^{\mathrm{P}}} w_j \geq |\mathcal{O}|$. Thus $|\mathcal{M}| \geq |\mathcal{O}|$ and by Lemma 1 the proof is complete. $\square$

We now give a lower bound on the capacity efficiency ratio of LDF by constructing a non-i.i.d. probabilistic adversary.

THEOREM 3. *Assume LDF is associated with the earliest-deadline-first (EDF) inter-link tie-breaking rule. There exists a probabilistic non-i.i.d. traffic process and deficit arrival process such that*

$$K_{\mathrm{LDF}} \geq \frac{\tau_{\max}}{\tau_{\min}}.$$

PROOF. For ease of presentation we only demonstrate a traffic process and a deficit arrival process with $\tau_{\max} = 3$ and $\tau_{\min} = 1$. General cases can be similarly constructed. Consider a network with three collocated links; i.e., the three links interfere with each other. We want to construct a traffic process such that LDF with capacity 2 cannot guarantee the full QoS region. Define the following traffic frames:

- J_1 with one packet per link. All packets arrive at the same time. The delay bounds for the three packets on links 1, 2 and 3 are 1, 2 and 3, respectively. The deficits increase at links 2 and 3;
- J_2 with one packet per link. All packets arrive at the same time. The delay bounds for the three packets on links 1, 2 and 3 are 2, 3 and 1, respectively. The deficits increase at links 1 and 2;
- J_3 with one packet per link. All packets arrive at the same time. The delay bounds for the three packets on links 1, 2 and 3 are 3, 1 and 2, respectively. The deficits increase at links 1 and 3;
- J_4 with one packet on link 1, two packets on link 2 and two packets on link 3. All packets arrive at the same time. The delay bounds are all 1. Each arriving packet increases the deficit by one.

Assume the initial deficits are all equal and in each frame we have arrival and deficit increases first, and then schedule according to current deficits. Let the adversarial traffic J be such that each big frame consists of (J_1, J_2, J_3) with probability $1 - \epsilon$, and (J_1, J_2, J_3, J_4) with probability ϵ, for some $\epsilon \in (0, 1)$. Then on average the deficit arrival is

$$\lambda = (1-\epsilon)\begin{pmatrix}2\\2\\2\end{pmatrix} + \epsilon\begin{pmatrix}3\\4\\4\end{pmatrix} = \begin{pmatrix}2+\epsilon\\2+2\epsilon\\2+2\epsilon\end{pmatrix}$$

per big frame, while on average LDF with EDF inter-link tie-breaking rule and capacity 2 schedules deficits

$$s_{\mathrm{LDF}} = (1-\epsilon)\begin{pmatrix}2\\2\\2\end{pmatrix} + \epsilon\begin{pmatrix}2\\3\\3\end{pmatrix} = \begin{pmatrix}2\\2+\epsilon\\2+\epsilon\end{pmatrix}$$

per big frame. So the deficits will increase linearly under LDF. We note that the optimal policy with unit capacity schedules

$$s_{\mathrm{OPT}} = \begin{pmatrix}3+\frac{1}{3}\epsilon\\3+\frac{1}{3}\epsilon\\3+\frac{1}{3}\epsilon\end{pmatrix}$$

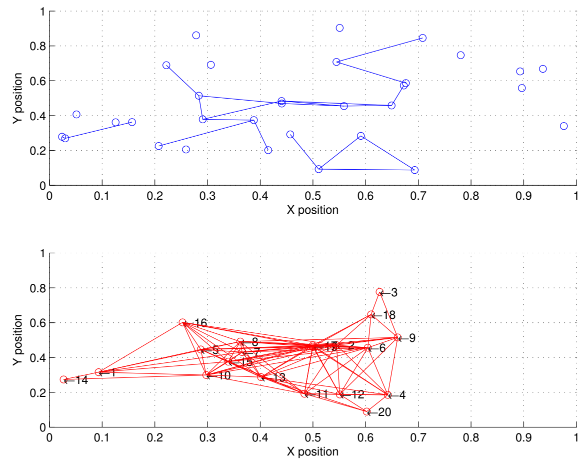

Figure 3: Network topology used in the simulations. The upper graph is the original graph and the lower graph is the conflict graph. The interference degree is 3.

per big frame. So for $\epsilon < 3/5$ we have

$$s_{\mathrm{LDF}} < \lambda < s_{\mathrm{OPT}}.$$

In other words, the capacity efficiency ratio for LDF with EDF tie-breaking rule under the given adversarial traffic process is at least $\tau_{\max}/\tau_{\min} = 3$. $\square$

4. SIMULATIONS

In this section we study the capacity efficiency ratio of different scheduling policies via simulations. We generated a randomly generated network shown in Fig. 3 with the unit-disk interference model.[5] In the original graph we place 32 nodes randomly in a unit square, and 20 links are randomly formed between pairs of nodes with distance less than the communication range $c = 0.25$. The interference is such that any two links interfere with each other if one transmitter or receiver is within the interference range $r = 0.21$ of the other transmitter or receiver. The number of packets arriving to each link at each time slot is $a_{\min} = 1$ with probability 0.5, and is $a_{\max} = 10$ otherwise. The delay bounds of the packets are $\tau_{\min} = 1$ for half random links, and $\tau_{\max} = 2$ for the other half. We varied some of the above default parameters in each specific simulation to show the impact they have on the system.

We compare LDF to two policies: the earliest-deadline-first policy with deficit-awareness (EDF-DA) and the shortest-delay-bound-first policy with deficit-awareness (SDBF-DA). The original versions of these two policies have been successfully used in real-time operating systems. EDF-DA is the QoS-guaranteeing version of the earliest-deadline-first scheduling algorithm, which is shown optimal on preemptive uniprocessors. It first gives priority to links with positive deficits, and then greedily choose packets with the earliest deadlines. SDBF-DA is the QoS-guaranteeing version of the rate-monotonic scheduling algorithm. It first gives priority

[5]The unit-disk model was first proposed to model ad hoc wireless networks by Huson and Sen [7], and later widely used in the analysis of scheduling problems in ad hod wireless networks (see, e.g., Joo et al. [10]).

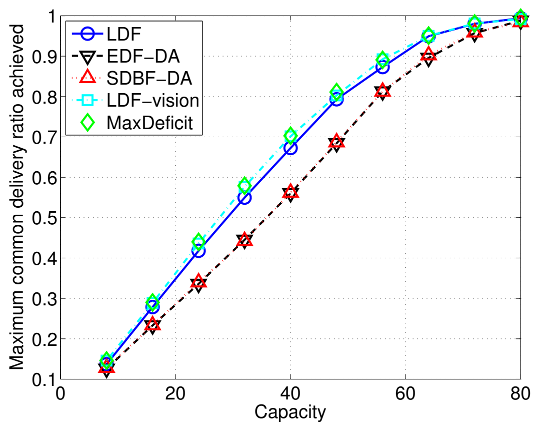

Figure 4: Maximum common delivery ratio achieved with different capacities for $\tau_{\max} = 1$.

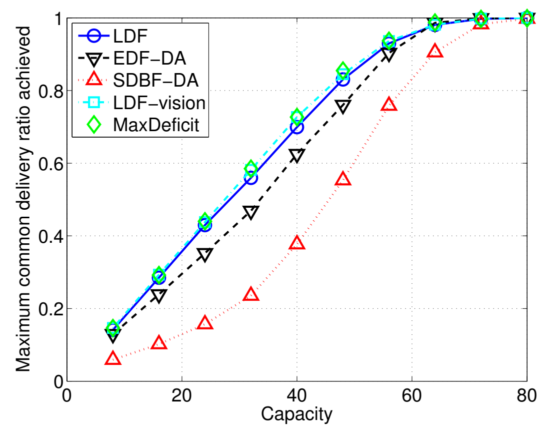

Figure 5: Maximum common delivery ratio achieved with different capacities for $\tau_{\max} = 2$.

to links with positive deficits, and then greedily choose packets with the shortest delay bound. Ties are broken uniformly at random.

We also consider two modified versions of LDF. The first one is LDF-vision, which starts with an empty selection, and in each repetition adds one or two feasible links to the selection in a greedy fashion such that the total deficit in the selection is maximized. We note that LDF-vision is a less greedy version of LDF, which greedily adds a single link to the selection. Intuitively LDF-vision has more global vision than LDF and may end up with a selection with larger total deficit. The other modified version of LDF is MaxDeficit, which gives the feasible schedule on the set of available links with the maximum total deficit. We can see that MaxDeficit is a max-weight scheduling with weight being the deficit. We note that MaxDeficit may incur much higher computational overhead than its greedy counterparts as the network size becomes large.

Once a link is chosen by the scheduler, SDBF-DA will schedule a packet with the shortest delay bound on that link, while all other policies will schedule a packet with the earliest deadline on that link.

Remark 2. *Note that since the optimal policy for the delivery ratio is unknown, the capacity efficiency ratio cannot be calculated. So in the simulations, we plotted the delivery ratio versus the capacity for each low-complexity algorithm to study the performance improvement of each when additional capacity is added to the network.*

4.1 Homogeneous Delivery Ratios

We let the delivery ratios for all links be equal to a common delivery ratio $\bar{q}$ and study the maximum common delivery ratio value that each policy can achieve with various capacity. The results are shown in Figure 4 for homogeneous delay bounds, and Figure 5 for heterogeneous delay bounds. We see that the marginal delivery rate improvement brought by adding capacity decreases significantly as the capacity becomes large. For example in Figure 4 an additional amount of 8 units of capacity increases the maximum QoS of LDF from 0.45 to 0.59 at capacity 24, but only 0.95 to 0.98 at capacity 64. This phenomenon, known as *diminishing returns* in economics, demonstrate that it could be very demanding to achieve high delivery ratio of one by boosting capacity due to occasional bursty arrivals.

The comparison between different policies in both figures show that LDF has better delivery ratio performance than EDF-DA and SDBF-DA. While in the homogeneous delay bound case in Figure 4 EDF-DA and SDBF-DA are identical since all available packets have the same deadlines and delay bounds, in the heterogeneous delay bound case in Figure 5 EDF-DA outperforms SDBF-DA. We conjecture that LDF outperforms the two algorithms because it takes into account the full deficit information, as opposed to the other two policies which only use partial deficit information (whether the deficit is positive or not). We also note that SDBF-DA has a degraded performance for heterogeneous delay bounds since new packets with small delay bound may block packets with large delay bound that arrived in the previous time slots, leading to bad delivery ratio performance for links with large delay bounds.

Both LDF-vision and MaxDeficit have slightly better performance than LDF in the homogeneous scenario. This is intuitively because LDF-vision and MaxDeficit usually give scheduling decisions with larger total deficit. It should be noted that LDF-vision and MaxDeficit are much more computational expensive than LDF. Further comparison between LDF and its two modified versions on stability region will be provided in Section 4.2.

4.2 Heterogeneous Delivery Ratios

Now we randomly divide the links in the network into two groups: Group A and Group B. We let each group use a common delivery ratio, and plot the boundary delivery-ratio pairs where the deficits start to become unstable in Figure 6 and Figure 7. We first notice that LDF achieves the largest delivery ratio stability region among the three deficit-aware policies. We also note that the performance of the different policies change significantly when the heterogeneity of the delivery ratio increases. In particular the delivery ratio stability regions of EDF-DA and SDBF-DA are not convex. Comparing Figure 6 with Figure 7 we again see that heterogeneity in delay bounds benefits LDF and EDF-DA but harms SDBF-DA.

Comparing LDF and its two modified versions, namely LDF-vision and MaxDeficit, we see that the delivery ratio

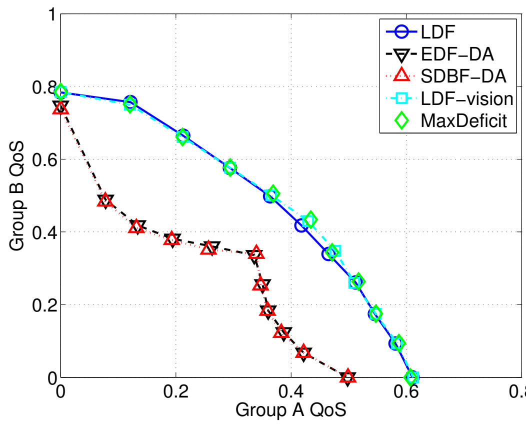

Figure 6: Delivery ratio stability region for $\tau_{\max} = 1$ with capacity 24.

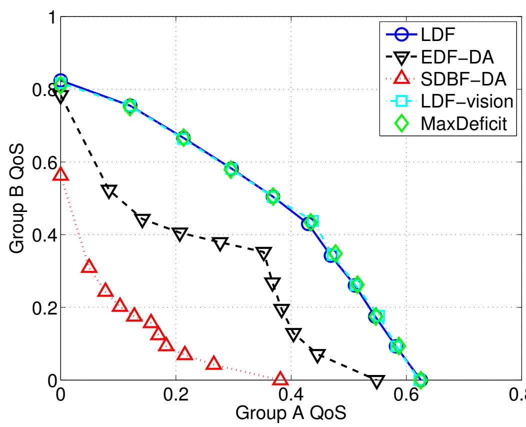

Figure 7: Delivery ratio stability region for $\tau_{\max} = 2$ with capacity 24.

stability regions for the three LDF-based policies are almost identical. The only noticeable difference between LDF and its modified versions is on the diagonal, i.e., the homogeneous case when the two groups have the same QoS. As a result, the computational advantage and the potential to be implemented distributedly makes LDF a good candidate for scheduling real-time traffic in ad hoc networks.

5. CONCLUSIONS

In this paper we studied the capacity requirements of ad hoc wireless networks with real-time traffic under the largest-deficit-first scheduling policy. We looked at the capacity efficiency ratio which is the amount of capacity needed for LDF to outperform the optimal policy with unit capacity. We derived upper and lower bounds on the capacity efficiency ratio of LDF in terms of max/min packet delay bound ratio and the interference degree of the conflict graph. We also used simulations to demonstrate the diminishing returns phenomenon of capacity.

6. ACKNOWLEDGMENTS

This work was supported in part by the National Science Foundation (NSF) under Grants CNS-1264012 and CNS-1262329, and the Defense Threat Reduction Agency (DTRA) under Grant HDTRA1-13-1-0030.

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APPENDIX

A. PROOF OF LEMMA 1

Let

$$C_{\mathrm{MAX}} = \min\{k\colon \mu^{[k]}(J,T) \geq \nu^{[1]}(J,T) \quad \forall J \in \mathcal{J}, \forall T, \forall \mu^{[k]} \in \mathcal{P}(k), \forall \nu^{[1]} \in \mathcal{P}(1)\},$$

We only need to show that

$$\mathrm{int}(\Lambda) \subseteq \Lambda_{\mathrm{LDF}^{[k]}}$$

for $k = C_{\mathrm{MAX}}$. Let $X(F)$ be the F-framed version of the traffic process X, which consists of all the packets j in X with $zF \leq b(j) \leq e(j) \leq (z+1)F$ for some integer z. Let the delivery ratio stability region with unit capacity for the F-framed traffic $X(F)$ be $\Lambda^{(F)}$. By definition of the stability region, for any $q \in \Lambda$ there exists a maximal scheduling policy with unit capacity, denoted by μ, that achieves q. Let $\mu^{(F)}$ be the F-framed version of μ, which only schedules packets in $X(F)$. Then

$$\mu_i^{(F)}(X,T) \geq \mu_i(X,T) - \frac{\tau_{\max} T}{F}.$$

Then with probability one,

$$\liminf_{T\to\infty} \frac{\mu_i^{(F)}(X,T)}{\mathrm{CA}_i(X(F),T)} \geq \liminf_{T\to\infty} \frac{\mu_i(X,T) - \frac{\tau_{\max}T}{F}}{\mathrm{CA}_i(X,T)} \geq q_i - \frac{\tau_{\max}}{F\bar{a}_i},$$

where $\bar{a}_i$ is the arrival rate of traffic X on link i. Then we can see that the delivery ratio stability region for the F-framed traffic $X(F)$ approaches Λ as $F \to \infty$. So for any $q \in \mathrm{int}(\Lambda)$ there exists F such that $q \in \Lambda^{(F)}$.

Let $\mathrm{MW}(F)$ be the scheduling policy that does a maximum weight schedule in each frame with size F. The weights are the deficits updated at the beginning of each frame. While this policy requires knowledge of the traffic in the whole frame at the beginning of each frame, it has been shown that this policy achieves the optimal delivery ratio region [9].

We now make use of a quadratic Lyapunov function and the above MaxWeight scheduling policy to show the stability of deficit process of LDF given a fixed $q \in \mathrm{int}(\Lambda)$ and capacity $k = C_{\mathrm{MAX}}$. Let $\Psi^{\mathrm{LDF}}(t)$ be the queue state of the LDF policy at time t, which includes the number of packets and their delay bounds on each link. Let $D^{\mathrm{LDF}}(t)$ be the deficit vector for LDF at time t. Note that $\left((D^{\mathrm{LDF}}(t), \Psi^{\mathrm{LDF}}(t))\colon t \geq 0\right)$ is a Markov chain. Let the Lyapunov function defined on deficit vector d and queue state ψ be

$$V(d,\psi) = \sum_i d_i^2.$$

Then

$$\begin{aligned}
& V\left(D^{\mathrm{LDF}}(t+F), \Psi^{\mathrm{LDF}}(t+F)\right) - V\left(D^{\mathrm{LDF}}(t), \Psi^{\mathrm{LDF}}(t)\right) \\
&= \sum_i \left(D_i^{\mathrm{LDF}}(t+F)\right)^2 - \sum_i \left(D_i^{\mathrm{LDF}}(t)\right)^2 \\
&= \sum_i \sum_{t'=t}^{t+F-1} \left(\left(D_i^{\mathrm{LDF}}(t'+1)\right)^2 - \left(D_i^{\mathrm{LDF}}(t')\right)^2\right) \\
&= \sum_i \sum_{t'=t}^{t+F-1} \left(\left(\left(D_i^{\mathrm{LDF}}(t') + A_i^{\mathrm{D}}(t') - S_i^{\mathrm{LDF}}(t')\right)^+\right)^2 - \left(D_i^{\mathrm{LDF}}(t')\right)^2\right) \\
&\leq \sum_i \sum_{t'=t}^{t+F-1} \left(2D_i^{\mathrm{LDF}}(t')\left(A_i^{\mathrm{D}}(t') - S_i^{\mathrm{LDF}}(t')\right) + \left(A_i^{\mathrm{D}}(t') - S_i^{\mathrm{LDF}}(t')\right)^2\right) \\
&\leq \sum_i \sum_{t'=t}^{t+F-1} 2(D_i^{\mathrm{LDF}}(t) + Fa_{\max})A_i^{\mathrm{D}}(t') - \sum_i \sum_{t'=t}^{t+F-1} 2(D_i^{\mathrm{LDF}}(t) - Fk)S_i^{\mathrm{LDF}}(t') + FN\left(a_{\max}^2 + k^2\right) \\
&\leq \sum_i \sum_{t'=t}^{t+F-1} 2D_i^{\mathrm{LDF}}(t)A_i^{\mathrm{D}}(t') - \sum_i \sum_{t'=t}^{t+F-1} 2D_i^{\mathrm{LDF}}(t)S_i^{\mathrm{LDF}}(t') + FN\left(a_{\max}^2 + k^2\right) + 2F^2N\left(a_{\max}^2 + k^2\right),
\end{aligned}$$

where $A_i^{\mathrm{D}}(t)$ is the deficit arrival on link i in time slot t, and $S_i^{\mathrm{LDF}}(t)$ is the deficit departure of LDF on link i in time slot t. For any deficit vector d and two subsets of the links $\mathcal{L}_1, \mathcal{L}_2 \subseteq \mathcal{N}$, we say $\mathcal{L}_1$ is *dominating in* $\mathcal{L}_2$ *under deficit* d if (1) $\mathcal{L}_1$ is nonempty, (2) $\mathcal{L}_1 \subseteq \mathcal{L}_2$, and (3) for any $i \in \mathcal{L}_1$ and any $j \in \mathcal{L}_2$,

$$d_i > d_j + F(k + a_{\max}).$$

By convention any nonempty subset of links $\mathcal{L} \subseteq \mathcal{N}$ is dominating in itself regardless of the deficit. We can then define the *smallest dominating set in subset* $\mathcal{L}$ *under deficit* d to be

$$\mathrm{SDS}(\mathcal{L}, d) = \{\mathcal{L}_1 \subseteq \mathcal{L}\colon \mathcal{L}_1 \text{ is dominating in } \mathcal{L} \text{ under } d\}.$$

Then for $D^{\mathrm{LDF}}(t)$ we let the first dominating group be

$$G_1(t) = \mathrm{SDS}(\mathcal{N}, D^{\mathrm{LDF}}(t)),$$

and define the first group deficit to be

$$\tilde{D}_{G_1}(t) = \max_i D_i^{\mathrm{LDF}}(t).$$

If the jth dominating group $G_j(t) \neq \mathcal{N}$ for some $j \geq 1$ we can then define the $(j+1)$st dominating group to be

$$G_{j+1}(t) = G_j(t) \cup \mathrm{SDS}(\mathcal{N}\backslash G_j(t), D^{\mathrm{LDF}}(t)),$$

and define the $(j+1)$st group deficit to be

$$\tilde{D}_{G_{j+1}}(t) = \max_{i \in \mathcal{N}\backslash G_j(t)} D_i^{\mathrm{LDF}}(t).$$

We note that $G_j(t) \subsetneq G_{j+1}(t)$. Since $\mathcal{N}$ is a finite set, there exists an integer $m \leq N$ such that $G_m(t) = \mathcal{N}$. Also by convention we let $G_0(t)=\emptyset$ and $\tilde{D}_{G_{m+1}}(t)=0$. Then we have

$$\emptyset = G_0(t) \subsetneq G_1(t) \subsetneq G_2(t) \subsetneq \cdots \subsetneq G_m(t) = \mathcal{N}.$$

By the definition of C_{MAX} we have for any deficit vector d, any queue initial state ψ, any traffic instance from t to $t+F-1$, and any subset of links $\mathcal{L}$ that have strictly larger deficits than any link in $\mathcal{N}\setminus\mathcal{L}$ throughout the whole frame $\{t, t+1, \ldots, t+F-1\}$, if $D^{\text{LDF}}(t)=D^{\text{MW}(F)}(t)=d$, $\Psi^{\text{LDF}}(t)=\Psi^{\text{MW}(F)}(t)=\psi$, then

$$\sum_{i\in\mathcal{L}}\sum_{t'=t}^{t+F-1} S_i^{\text{LDF}}(t') \ge \sum_{i\in\mathcal{L}}\sum_{t'=t}^{t+F-1} S_i^{\text{MW}(F)}(t')$$

where $S_i^{\text{MW}(F)}(t)$ is the deficit departure of $\text{MW}(F)$ on link i in time slot t. So,

$$\begin{aligned}
&\sum_i D_i^{\text{LDF}}(t)\sum_{t'=t}^{t+F-1} S_i^{\text{LDF}}(t') \\
&= \sum_{j=1}^{m}\sum_{i\in G_j(t)\setminus G_{j-1}(t)} D_i^{\text{LDF}}(t)\sum_{t'=t}^{t+F-1} S_i^{\text{LDF}}(t') \\
&\ge \sum_{j=1}^{m}\sum_{i\in G_j(t)\setminus G_{j-1}(t)} \Bigg(\left(\tilde{D}_{G_j}(t) - NF(k+a_{\max})\right) \\
&\quad \cdot \sum_{t'=t}^{t+F-1} S_i^{\text{LDF}}(t')\Bigg) \\
&\ge \sum_{j=1}^{m}\tilde{D}_{G_j}(t)\sum_{i\in G_j(t)\setminus G_{j-1}(t)}\sum_{t'=t}^{t+F-1} S_i^{\text{LDF}}(t') \\
&\quad - F^2N^2(a_{\max}+k)k \\
&= \sum_{j=1}^{m}\left(\tilde{D}_{G_j}(t)-\tilde{D}_{G_{j+1}}(t)\right)\sum_{i\in G_j(t)}\sum_{t'=t}^{t+F-1} S_i^{\text{LDF}}(t') \\
&\quad - F^2N^2(a_{\max}+k)k \\
&\ge \sum_{j=1}^{m}\left(\tilde{D}_{G_j}(t)-\tilde{D}_{G_{j+1}}(t)\right)\sum_{i\in G_j(t)}\sum_{t'=t}^{t+F-1} S_i^{\text{MW}(F)}(t') \\
&\quad - F^2N^2(a_{\max}+k)k \\
&= \sum_{j=1}^{m}\sum_{i\in G_j(t)\setminus G_{j-1}(t)}\tilde{D}_{G_j}(t)\sum_{t'=t}^{t+F-1} S_i^{\text{MW}(F)}(t') \\
&\quad - F^2N^2(a_{\max}+k)k \\
&\ge \sum_i D_i^{\text{MW}(F)}(t)\sum_{t'=t}^{t+F-1} S_i^{\text{MW}(F)}(t') \\
&\quad - F^2N^2(a_{\max}+k)k.
\end{aligned}$$

Then by Jaramillo et al. [9],

$$\begin{aligned}
&\mathbb{E}\Big[V\left(D^{\text{LDF}}(t+F), \Psi^{\text{LDF}}(t+F)\right) \\
&\quad - V\left(D^{\text{LDF}}(t), \Psi^{\text{LDF}}(t)\right) \\
&\quad \Big| D^{\text{LDF}}(t)=d, \Psi^{\text{LDF}}(t)=\psi\Big] \\
&\le \mathbb{E}\left[\sum_i 2d_i\sum_{t'=t}^{t+F-1} A_i^{\text{D}}(t') \,\middle|\, D^{\text{MW}(F)}(t)=d, \Psi^{\text{MW}(F)}(t)=\psi\right] \\
&\quad - \mathbb{E}\left[\sum_i 2d_i\sum_{t'=t}^{t+F-1} S_i^{\text{MW}(F)}(t') \,\middle|\, D^{\text{MW}(F)}(t)=d, \Psi^{\text{MW}(F)}(t)=\psi\right] \\
&\quad + FN\left(a_{\max}^2+k^2\right) + 2F^2N\left(a_{\max}^2+k^2\right) \\
&\quad + F^2N^2(a_{\max}+k)k \\
&\le B_1 - B_2\sum_i d_i.
\end{aligned}$$

Then by the Foster–Lyapunov theorem the Markov chain $\left((D^{\text{LDF}}(t), \Psi^{\text{LDF}}(t))\colon t\ge 0\right)$ is positive recurrent. So the delivery ratio vector q is achieved by LDF. Hence $K_{\text{LDF}} \le C_{\text{MAX}}$.